Categories and Sheaves

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

Since their introduction by Eilenberg and Mac Lane [3], categories have been used to generalise the structures of mathematics. In this thesis we discuss categories of groups, topological spaces, and sets and we will study the properties they and do not share. We discuss the definitions of categories, functors, natural transformations, limits and colimits. For each of these definitions we will consider both familiar and more abstract examples as well as the theorems that relate them. Finally we will use category theory to introduce a special type of functor called a sheaf. A description of sheaf construction through the étalé space will be given, which in turn allows us to explore the connectedness of topological spaces.

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Contents

Chapter 1

What is a Category?

1.1 Defining a Category

In mathematics we study objects such as sets, rings or group, vector spaces, topologies, and algebras. For each of these objects we specify specific maps between them like homomorphisms or continuous maps. One might notice that these objects and their maps share properties, for example with most of these objects we can define a kernel of a map between them. The study of these objects, their similarities and their differences, is what we call category theory. The property of having special types of maps between these objects is the starting point for studying category theory.

Definition 1.1.1. A ***category*** C is a class $\mathrm{ob}(C)$ of ***objects*** and a class $\hom(C)$ of ***morphisms*** between the objects such that:

1. For every object X there exists the ***identity morphism*** $1_X : X \to X$ from X to X.
2. For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$, there exists a ***composite morphism*** $gf : X \to Z \in \hom(C)$.
3. For every morphism $f : X \to Y$, the composition satisfies $1_Y f = f 1_X = f$.
4. Composition of any three morphisms $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ is associative. That is $h(gf) = (hg)f = hgf$.

Furthermore if f is a morphism $f : X \to Y$ from object X to object Y then X is the ***domain*** of f and Y is the ***codomain*** of f. We can see that the objects we have already mentioned are categories once we describe their morphisms. Some examples are as follows.

Example 1.1.2.

1. Sets are objects with functions between them as morphisms. The identity map for each set is its identity morphism, composition of functions is well defined and is associative. We denote this category **Set**.

2. Rings are objects with the morphisms of homomorphisms. Homomorphisms are a type of function so clearly satisfy the axioms. We denote this category **Ring**.

3. Commutative rings give us the category **CRing** with morphisms as ring homomorphisms.

4. The category **Grp** has objects as groups and the morphisms are group homomorphisms.

5. Abelian groups and their group homomorphisms form the category **Ab**.

6. Finite vector spaces over a field K are objects with the morphism as linear maps. They form the category $\mathbf{Vec}_K$.

7. The class of topologies as objects with continuous functions as morphisms form a category. We denote this category **Top**.

8. The category of based point topologies has the objects (X, x) where X is a topology and $x \in X$ is the base point. Base point preserving continuous functions are the morphisms of this category which is denoted $\mathbf{Top}_*$.

9. The class of rings with morphisms as maps of the underlying sets is a category.

Where the morphisms are obvious we will often refer to a category just by its objects.

Definition 1.1.3. Let C and D be categories. We say that D is a ***subcategory*** of C denoted $D \subseteq C$ if every object of D is an object of C and every morphism of D is a morphism of C.

A similarly familiar structure is that of products of categories.

Definition 1.1.4. The product of two categories C and D defines a ***product category*** $C \times D$. A unique object $X \times Y \in \text{ob}(C \times D)$ is given for every $X \in \text{ob}(C)$ and $Y \in \text{ob}(D)$. The morphisms $f : X \times Y \to Z \times W \in \text{hom}(C \times D)$ are given uniquely as the pair $f = (f_C, f_D)$ for every pair of morphisms $f_C \in \text{hom}(C)$ and $f_D \in \text{hom}(D)$ where we define composition component-wise and $(1_X, 1_Y)$ is the identity on the object $X \times Y$.

Despite the complexity of some of these categories, the definition of a category requires only a little structure. In fact our definition fundamentally says that a category can be described by some collection of points and arrows between them. This means that unlike most of our examples we do not actually need a set structure for our objects at all. An easy example of a category where we can do this is the poset on the natural numbers.

Example 1.1.5. A poset will always form a category. Take objects as the elements of the poset category. The morphisms of this category exist uniquely between objects $x \to y$ exactly when $x \geq y$. Reflexivity of $\geq$ guarantees the existence of the identity morphisms and transitivity gives us the correct composition. The class of natural numbers as objects together with the $\leq$ relation form a category. We can describe this category with the following graph,

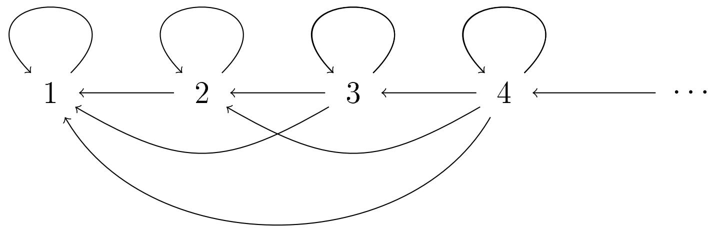


Figure 1.1: Graph of the poset category on $\mathbb{N}$.

When describing a category like this the identity morphism will be implicit and we will not draw the loops. Similarly if composition of morphisms commute as in the case for a poset we will often not add these morphisms explicitly. Instead, they are implied by the composition of the morphisms we have described. To simplify the graph of Figure 1.1 in this way would give us the graph in Figure 1.2.

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow \cdots$$

Figure 1.2: Simplified graph of poset category on $\mathbb{N}$.

Example 1.1.6. Posets do not necessarily have to be totally ordered. For example, Figure 1.3 describes a valid poset and so is a valid category.

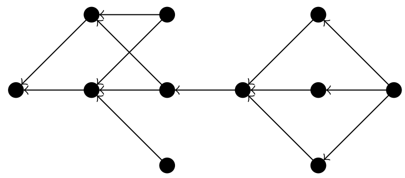


Figure 1.3: Some arbitrary poset category.

Remark 1.1.7. A poset as a category is **not** the same as the category **Poset** of partially ordered sets and order preserving morphisms.

Example 1.1.8. We can use this idea of using graphs to generate categories to define arbitrary. The graphs given by a category are called ***quivers***. A simple collection of categories are the ***discrete categories***. These are categories whose only morphisms are the identity morphisms. Finite discrete categories can be represented as a graph of n nodes with no edges.

Figure 1.4: Finite discrete category.

For a category C we also define the ***opposite category*** C^{op}. This is the category defined as having the same objects as C and but the morphisms of C^{op} are those of C but with the domain and co-domain swapped. The category C^{op} retains the same identity morphisms and the compositions are well defined because they are well defined in C. For example, the opposite category for the poset described in Figure 1.1 is defined by morphisms existing from x to y when $x \leq y$. For the quiver of a category we can simply reverse the directions of the vertices to represent the opposite category.

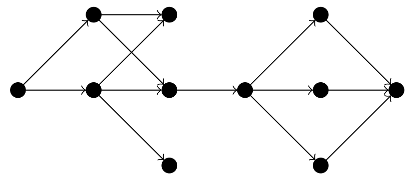


Figure 1.5: Opposite category of Figure 1.3.

In addition to the discrete and poset categories, there are other useful and interesting abstract categories.

Example 1.1.9.

1. A group G defines a single object category denoted BG where the morphisms are elements of G and their composition is multiplication. Each of these group elements are morphisms from the group to itself.
2. For a unital ring, the category denoted $\mathbf{Mat}_R$ has strictly positive integers for its objects. The morphisms $m \to n$ are $n \times m$ matrices over R with composition defined by left multiplication.
3. The class of topological spaces can give rise to the homotopy category as well as the category of topologies mentioned earlier. In this category the morphisms are the homotopy classes of continuous maps with the direction of the morphisms in the same direction as said maps.

1.2 Types of Morphism

In the case of the categories, like **Set**, in Example 1.1.2 we have some of notion of isomorphism where two objects are considered the same by the existence of an invertible morphism between them. For example for groups and rings, if a homomorphism is bijective and the inverse is a homomorphism then we call the homomorphism an isomorphism and our two objects are isomorphic. We can easily generalise this notion to any category.

Definition 1.2.1. An ***isomorphism*** $f : X \to Y$ in a category is a morphism such that there exists a morphism $g : Y \to X$ where the compositions $fg = 1_Y$ and $gf = 1_X$. If an isomorphism exists between two objects X and Y then we say that X and Y are ***isomorphic***. In this case we write $X \cong Y$.

Much like how we can show the inverse of a group element is unique, the inverse of an isomorphism is unique.

Lemma 1.2.2. *Every isomorphism in a category has a unique inverse.*

Proof. Let $f : X \to Y$ be an isomorphism with inverses $g : Y \to X$ and $h : Y \to X$. By definition $fg = fh = 1_Y$ and so $hfg = hfh$ which implies $1_X g = 1_X h$ and in turn $g = h$. Hence, the inverse of f is unique. We denote this unique inverse as f^{-1}. □

Definition 1.2.3. A morphism from an object to itself is an ***endomorphism*** and if this is also an isomorphism we call it an ***automorphism***.

For some more examples of isomorphisms:

Example 1.2.4.

1. The isomorphisms of sets are the bijections.

2. The isomorphisms of topologies are the homeomorphisms.

3. The identity morphism of a category is always an automorphism.

4. All morphisms in the category BG are automorphisms. To prove this, notice that the identity element of G is the identity morphism of BG, and that every element of G has an inverse.

5. In the homotopy category (of topological spaces and the homotopy classes of continuous maps) the isomorphisms are the homotopy equivalences.

So we have a general notion of isomorphisms in our categories. We might ask if we are also able to generalise the concepts of injective and surjective functions. We can write a necessary and sufficient property for a function on sets to be surjective.

A function $f : X \to Y$ is surjective if and only if for all maps $g, h : Y \to Z$ such that $gf = hf$, we have $g = h$. This is because for gf to equal hf then they must both be the same on the image of f and so are equal whenever f is surjective. We will show the other direction by showing the contrapositive. That is, if $f : X \to Y$ is not a surjection, then there exists distinct functions $g, h : Y \to Z$ such that $gf = hf$. Take a point $y \in Y$ which is not in the image of f. Let $g : Y \to \{0, 1\}$ define the function which sends y to 1 and every other point to 0. Let $h : Y \to \{0, 1\}$ define the constant map which is 0 everywhere. Clearly we have $gf = hf = 0$ for all $x \in X$ but $g \neq h$.

We have a similar property for injective functions. A function $f : X \to Y$ is injective if and only if for all maps $g, h : Z \to X$ such that $fg = fh$, we have $g = h$. If f is injective then we can only have $fg = fh$ when $g = h$ because f has a unique inverse on every element of its image. Conversely, if f is not injective then there exists distinct points x and x' in X such that $f(x) = f(x')$. In this case we can have $g(z) = x$ and $h(z) = x'$ while still allowing $fg = fh$.

We take this property of surjective and injective functions on sets to define a new type of morphism in general categories which act like surjective and injective functions.

Definition 1.2.5. An ***epimorphism*** is a morphism $f : X \to Y$ such that for every pair of morphisms $g : Y \to Z$ and $h : Y \to Z$, if $gf = hf$ then $g = h$. This is a categorical analogue of surjectivity. We say a morphism is ***epic*** if it is an epimorphism.

Definition 1.2.6. An ***monomorphism*** is a morphism $f : X \to Y$ such that for every pair of morphisms $g : Z \to X$ and $h : Z \to X$, if $fg = fh$ then $g = h$. This is a categorical analogue of injectivity. We say a morphism is ***monic*** if it is an monomorphism.

Notice that in the opposite of a category these properties of a morphism are reversed; i.e. if $f : X \to Y \in C$ if an epimorphism then $f^{op} : Y \to X \in C^{op}$ is a monomorphism and similarly if f is monic then f^{op} is monic is an epimorphism.

Since epic and monic are categorical analogues of surjective and injective, it might be natural to ask how this relates to isomorphisms. Certainly in the category of sets a morphism is isomorphic if and only if it is monic and epic.

Lemma 1.2.7. *Every isomorphism is both monic and epic.*

Proof. Let $f : X \to Y$ be an isomorphism so that there exists f^{-1} such that $ff^{-1} = 1_Y$ and $f^{-1}f = 1_X$. If $fg = fh$ for some pair of morphisms $g : Z \to X$ and $h : Z \to X$ then $fg = fh$ implies $f^{-1}fg = f^{-1}fh$ and so simplifying gives us $g = h$. Similarly but with right composition if g, h are morphisms $Y \to Z$ then $gf = hf$ implies $g = h$ and so f is both monic and epic. □

Perhaps surprisingly, the reverse is not true.

Example 1.2.8. A simple example of a epic monic morphism which is not an isomorphism is in the category in Figure 1.6.

$$X \xrightarrow{\quad f \quad} Y$$

Figure 1.6: A monic epic morphism which is not isomorphic.

The morphism f between the two objects is monic and epic because the only functions $Y \to Z$ and $W \to X$ for some W, Z are the identity morphisms where we take $W = X$ and $Z = Y$. Now our condition for monomorphisms and epimorphisms hold trivially. However, since there is no morphism $Y \to X$ then there cannot be an inverse to f and so it is not an isomorphism.

Example 1.2.9. For a more natural example of a monic epic morphism which is not an isomorphism consider the inclusion morphism $i : \mathbb{Z} \to \mathbb{Q}$ in the category of rings. Since i is not surjective it cannot be a ring isomorphism, in fact there are no ring morphisms $\mathbb{Q} \to \mathbb{Z}$. To show that i is monic and epic consider the following proof.

Proof. For any ring R, let f and g be homomorphisms $R \to \mathbb{Z}$ such that $if = ig$ then for $x \in \mathbb{Z}$ we see $f(x) = i(f(x)) = i(g(x)) = g(x)$ as i is just the inclusion and hence i is monic. To see that i is epic let $f, g : \mathbb{Q} \to R$ for some ring R. Since f, g are ring homomorphisms and $f(n) = g(n)$ for integer n then:

$$\begin{aligned}
f\left(\frac{1}{n}\right) &= f\left(\frac{1}{n}\right)\left(g(n)g\left(\frac{1}{n}\right)\right) \\
&= \left(f\left(\frac{1}{n}\right)g(n)\right)g\left(\frac{1}{n}\right) \\
&= \left(f\left(\frac{1}{n}\right)f(n)\right)g\left(\frac{1}{n}\right) \\
&= 1_R g\left(\frac{1}{n}\right) \\
&= g\left(\frac{1}{n}\right)
\end{aligned}$$

Hence,

$$g\left(\frac{m}{n}\right) = g(m)g\left(\frac{1}{n}\right) = f(m)f\left(\frac{1}{n}\right) = f\left(\frac{m}{n}\right)$$

for every $\frac{m}{n} \in \mathbb{Q}$ and so i is indeed epic. $\square$

Chapter 2

Functors

2.1 Defining a Functor

As with lots of mathematics, categories become most useful when we start to describe relations between them. To do this we can define a map of categories.

Definition 2.1.1. A ***functor*** $F : C \to D$ of categories is a map $\mathrm{ob}(C) \to \mathrm{ob}(D)$ and a map $\hom(C) \to \hom(D)$ such that:

1. The domain and codomain of a morphism $f : X \to Y \in C$ after applying F are $F(X)$ and $F(Y)$ respectively.
2. For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in C then $F(gf) = F(g)F(f)$.
3. F preserves identity morphisms. That is for an object $X \in C$ we have $F(1_X) = 1_{F(X)}$.

We can consider some examples of functors.

Example 2.1.2.
1. For categories C, D and object $X \in D$, he constant functor $F_X : C \to D$ sends every object in C to X and every morphism in C to $\mathrm{id}_X \in D$.
2. The functor sending a set to the free group on its elements is a functor **Set** $\to$ **Grp**.
3. For two groups G, H then a functor $F : BG \to BH$ is precisely a group homomorphism $G \to H$.
4. The fundamental group π_1 of a based topological space is a functor $\mathbf{Top}_* \to$ **Grp**.
5. The identity functor $F : C \to C$ sends each object and morphism to themselves. A functor from a category to itself is an ***endofunctor***.
6. Another endofunctor is the power set functor F from the category of sets to itself such that each object X is sent to its power set $\mathcal{P}(X)$ and each morphism $f : X \to Y$ to the morphism $F(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ which sends $U \in \mathcal{P}(X)$ to its image $f(X) \in \mathcal{P}(Y)$.

7. Any functor $F : C \to D$ induces the ***opposite functor*** $F^{op} : C^{op} \to D^{op}$ which acts the same on the objects and morphisms of C^{op} as F does on C.

8. For group G and category C, if $\bullet$ is the object of BG, then a functor $F : BG \to C$ defines the ***left action*** of G on the object $F(\bullet) \in C$ by where F sends the group elements $g \in \hom(BG)$. If C is the category of sets then F decides how each element of g acts as an endomorphism of the set $F(\bullet)$. This endomorphism defines a left action on $F(\bullet)$ because the morphism composition is a left composition. When endowed with such an action $F(\bullet)$ is called a ***G-set***. If C is the category of vector spaces then $F(\bullet)$ with its action is a ***G-representation***.

A large class of functors are the ***forgetful functors***. These are functors which "forget" some of the properties or structure of the domain category.

Example 2.1.3.

1. The easiest type of forgetful functors are those on categories with set structure. A forgetful functor then forgets all properties except for the set structure. For example, there is a forgetful functor **Grp** $\to$ **Set** which sends each group to its underlying sets.

2. Another example of this type of functor is that from the category of topologies to the category of sets by taking each topological space to its underlying set.

3. Using a similar idea we have the forgetful functor from rings to the category of abelian groups which sends a ring to its additive group.

4. The inclusion functor from abelian groups to the category of all groups.

5. The functor from Lie groups to the category of differentiable manifold which "forgets" the group action and sends Lie group homomorphisms to their corresponding differentiable function is a forgetful functor. Similarly we can take a functor from Lie groups to the category of groups, forgetting the manifold structure.

At this point we already have sufficient tools to start showing interesting results which do not directly relate to category theory. The one I shall show now now is known as Brouwer's fixed-point theorem. This theorem may be familiar to the reader when described as in terms of the following real world analogue. If you place a piece of (possibly) crumpled paper on top of another identical piece then there is at least one coordinate on the crumpled paper which is directly above the same coordinate on the other sheet.

Theorem 2.1.4 (Brouwer Fixed Point Theorem)**.** *Every continuous function from a closed disk to itself has a fixed point.*

Proof. Assume for contradiction that a map $f : D^2 \to D^2$ exists without fixed points. If such an f exists then there also exists a retraction $r_f : D^2 \to S^1$ to the circle, we define this by taking, for every point $x \in D^2 \setminus S^1$, a line from x through $f(x)$ and defining $r_f(x)$ as the point where this line intersects the circle.

This is now where our category theory is useful. As mentioned in Example 2.1.2, there exists a functor π_1 from the category of based topological spaces to the category of groups by taking the fundamental group of the space. Take a base point on the boundary of the disk. By the definition of the retraction then the inclusion map $i : S^1 \to D^2$ composed with r is the identity $ri = \text{id}_{S^1}$ on the circle. Since $\pi_1(D^2) = \{0\}$ then $\pi_1(r) : \pi_1(D^2) \to \pi_1(S^1)$ must have an image of $\{0\}$ and so $\pi_1(r)\pi_1(i)$ must be the 0 function. We know that the fundamental group $\pi_1(S^1) = \mathbb{Z}$ and so using the definition of a functor we know π_1 must satisfy $\pi_1(r)\pi_1(i) = \pi_1(ri) = \pi_1(\text{id}_{S^1}) = \text{id}_{\pi_1(S^1)} = \text{id}_{\mathbb{Z}}$ and since $\text{id}_{\mathbb{Z}}(1) = 1 \neq 0$ then we have a contradiction. Hence, the retraction cannot exist and so neither can our continuous endomorphism. □

Now, to continue our discussion of functors, a collection of functors relevant to our discussion are the following:

Definition 2.1.5. A ***contravariant functor*** from category C to D is a functor $F : C^{op} \to D$. Explicitly this is different from a regular functor in that instead of axiom (1) we have:

1. The domain and codomain for a morphism $f : X \to Y \in C$ after applying F are $F(Y)$ and $F(X)$ respectively. That is $F(f)$ is a morphism $F(Y) \to F(X)$.

Due to this change we must also reverse the order of composition. That is, instead of axiom (2) we have:

2. For every pair of morphisms $f : X \to Y$ and $g : Y \to X \in C$ then $F(gf) = F(f)F(g)$.

We call a 'regular' functor ***covariant*** so as to not confuse them with contravariant functors. As with covariant functors there are plenty of natural examples of contravariant functors.

Example 2.1.6.

1. On the category of real vector spaces the map which sends every vector space to its dual and every linear map to its transpose is a contravariant endofunctor.

2. The functor $F : \text{Set}^{op} \to \text{Set}$ which sends a set to its power set and a morphism $f : X \to Y$ to its inverse $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$.

3. A contravariant functor $C \to C^{op}$ which is the identity on the objects of C is just the identity functor $C^{op} \to C^{op}$.

4. Where the covariant functor $BG \to \textbf{Set}$ defines the left action, the functor $F : BG^{op} \to \textbf{Set}$ defines a ***right action*** and F is a contravariant functor on BG.

5. A particularly important class of functors are ***presheaves***. A presheaf is a contravariant functor $F : C \to \textbf{Set}$. Often we take C to be the poset category of open sets in a topological space. We will discuss presheaves in more detail later.

It is worth noting that a contravariant functor $F : C \to D$ is just a covariant functor $F : C^{op} \to D$. We will describe 'a contravariant functor $C \to D$' as 'a functor $F : C^{op} \to D$' to avoid confusion in line with modern literature.

2.2 Category Size and Fully Faithful Functors

It is natural to ask that if we have functors between categories then surely this would allow all categories to themselves form a category. Strictly the answer is no for the same reason we cannot have a set of sets [2]. This is why in Definition 1.1.1 we describe a "class" of objects and a "class" of morphisms as opposed to sets. There are however several methods category theorists use to deal with this problem as described in [8]. An example is Bernays-Gödel set theory which lets us define classes of objects which are larger than sets. In general this is not something that should be of concern for our purposes but a useful notion to help avoid these issues is the following.

Definition 2.2.1. A category C is ***small*** if $\hom(C)$ contains only a set's worth of elements.

There's an obvious map $\hom(C) \to \text{ob}(C)$ which takes a morphism to its domain. Notably $\text{id}_X \mapsto X$ under this map so this is a surjection and hence a small category will contain only a set's worth of objects. By limiting the size of a category we there are no issues, for example, in defining the category of small categories where the morphisms are functors.

We can also consider size of the class of a morphism between two objects of a category.

Definition 2.2.2. A category is ***locally small*** if between every pair of objects there is only a set's worth of morphism in which case we write $\hom(X, Y)$ to denote the set of morphisms between X and Y. Such a set is called a ***hom set***.

By introducing locally small categories we can give an important example of both a covariant and contravariant functor.

Example 2.2.3. Let C be a category. Define a functor $\hom(X, -) : C \to \text{Set}$, which takes an object $Y \in C$ to the set of morphisms $\hom(X, Y)$ and takes each morphism $f : Y \to Z$ to the morphism denoted $f_* := \hom(X, f)$ defined on an element $g \in \hom(X, Y)$ as the left composition $f_*(g) = f \circ g : X \to Z$. Similarly, define the functor $\hom(-, X) : C^{op} \to \text{Set}$ by sending an object $Y \in C$ to the set of morphisms $\hom(Y, X)$ and takes each morphism $f : Z \to Y$ to the morphism $f^* := \hom(f, X)$ defined on $g \in \hom(Y, X)$ as the right composition $f^*(g) = g \circ f : Z \to X$. We call these functors ***hom functors***.

Now we have some ways to describe properties of categories, let us see how functors behave. An important property of functors is given by the following lemma.

Lemma 2.2.4. *Functors preserve isomorphisms.*

Proof. Let $F : C \to D$ be our functor. If $f : X \to Y \in C$ is an isomorphism then,

$$F(f)F(f^{-1}) = F(ff^{-1}) = F(\mathrm{id}_Y) = \mathrm{id}_{F(Y)}$$

and the similarly $F(f^{-1})F(f) = \mathrm{id}_{F(X)}$. □

Unfortunately functors do not preserve monomorphisms or epimorphisms.

Example 2.2.5. The forgetful functor $F : \mathbf{Ring} \to \mathbf{Set}$ does not preserve epimorphisms. The inclusion $\iota : \mathbb{Z} \to \mathbb{Q}$ is epic as mentioned previously but $F(\iota)$ is not because the inclusion $\mathbb{Z} \to \mathbb{Q}$ as sets is not surjective. By symmetry the same corresponding morphism in $\mathbf{Ring}^{op}$ is monic but this is not preserved by the functor F^{op}.

To make matters worse, while $X \cong Y \implies F(X) \cong F(Y)$ the converse is not true in general. For that we need a new descriptor of functors.

Definition 2.2.6. Let $F : C \to D$ be a functor where C and D are locally small categories. The functor F is said to be ***full*** if for all objects $X, Y \in C$ the function $\hom(X, Y) \to \hom(F(X), F(Y))$ induced by F is surjective. If the function induced by F is injective then F is ***faithful***. If it is bijective, then F is said to be ***fully faithful***.

This condition allows for the following lemma.

Lemma 2.2.7. *If $F : C \to D$ is a fully faithful functor and X, Y are objects in C then $F(X)$ is isomorphic to $F(Y)$ if and only if $X \cong Y$.*

Proof. As has been show, the backwards direction is true of all functors. It remains to see the converse is true for fully faithful functors. Let $F : C \to D$ be fully faithful and $F(X) \cong F(Y)$ be isomorphic in D. Since $F_{X,Y} : \hom(X, Y) \to \hom(F(X), F(Y))$ is bijective in particular it is surjective and there exists and $f \in C$ such that $F(f) : F(X) \to F(Y)$ is an isomorphism. Since F is full, there exists $g \in C$ is such that $F(g) = F(f)^{-1}$. It remains to show that $g = f^{-1}$. Since $F(f)$ is a morphism $F(X) \to F(Y)$ and g is a morphism $F(Y) \to F(X)$ then by the definition of a functor we know the domains and codomains, $f : X \to Y$, $g : Y \to X$ and in particular fg, gf are both well defined. Notice then that $F(fg) = F(f)F(g) = \mathrm{id}_{F(Y)}$ but since F is faithful there is only one morphism h such that $F(h) = \mathrm{id}_{F(Y)}$ and since the identity id_Y must satisfy this property we have $fg = h = \mathrm{id}_Y$. Similarly $gf = \mathrm{id}_X$ and so $g = f^{-1}$ as required. □

Chapter 3

Natural Transformations

3.1 Defining a Natural Transformation

So we have functors as a tool to map between categories. Natural transformations take this a step further as a map between functors themselves.

Definition 3.1.1. Let $F, G : C \to D$ be functors. A ***natural transformation*** $\eta : F \to G$ is a collection of morphisms of D denoted η_X for each object $X \in C$ such that for every morphism $f : X \to Y$ the following diagram commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

That is, for all X, Y and morphism $f : X \to Y$ we have $\eta_Y F(f) = G(f)\eta_X$. We call the morphisms η_X the ***components*** of η and if every component of η is an isomorphism of D then we say η is a ***natural isomorphism*** and $F \cong G$.

As with the case of functors and categories before that, the role of natural transformations is best illustrated with some examples.

Example 3.1.2.

1. Take a fixed $n \in \mathbb{N}$ and consider the functor $\mathrm{GL}_n : \mathbf{CRing} \to \mathbf{Grp}$ which sends R to $n \times n$ matrices over R and a morphism f in $\mathbf{CRing}$ to the morphism f on each element of a matrix in GL_n. Let $G : \mathbf{CRing} \to \mathbf{Grp}$ be a functor which sends a ring to its group of units $R^\times$. The determinant is a natural transformation $\mathrm{GL}_n \to G$. The determinant will always send a

member of GL_n to $R^\times$ because if $A \in R$ then $\det(A)^{-1} = \det(A^{-1})$ and by definition $A^{-1} \in \mathrm{GL}(R)$. Because the determinant is a polynomial defined the same for each matrix and R is commutative then det commutes with f and $\mathrm{GL}_n(f)$ as required.

2. A simpler example takes GL_n : **CRing** $\to$ **Grp**, defined as above, to the forgetful functor, F : **CRing** $\to$ **Grp**. Define the natural transformation as to take the top left entry of $\mathrm{GL}_n(R)$ to the same element of its ring.

3. We saw in Example 2.1.6 that the dual of a vector space is a contravariant endofunctor. This means the double dual is a covariant endofunctor. For each vector space V there is a injective morphism η_V which sends $v \in V \mapsto v \in V^{**}$ and these form a natural transformation because they are each injective so the diagram commutes. In fact when the dimension of V is finite this is a natural isomorphism because $V \cong V^{**}$ when $\dim(V) < \infty$.

4. There is a natural transformation η from the identity functor on sets to the covariant power set functor P which sends a set to its power set. For set X then η_X sends $x \in X$ to $\{x\} \in \mathcal{P}(X)$. For $f : X \to Y$ we clearly have $\eta_Y f = P(f)\eta_X$ and so this collection gives a natural transformation.

5. For a group G the commutator subgroup denoted $[G, G]$ is the normal subgroup generated by elements of the form $g^{-1}h^{-1}gh$. This allows the ***abelianization*** $G \to G_{\text{ab}}$ of a group by sending G to the quotient group $G/[G, G]$. Abelianization is a functor A : **Grp** $\to$ **Ab** which sends a homomorphism to its corresponding morphism on G_{ab}. The collection of projections $\pi_G : G \to G_{\text{ab}}$ from a group G to its cosets are the components of a natural transformation $\pi : \mathrm{id}_{\mathbf{Grp}} \to A$. The diagram,

$$\begin{array}{ccc} G & \xrightarrow{\pi_G} & G_{\text{ab}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle A(f)} \\ H & \xrightarrow[\pi_H]{} & H_{\text{ab}} \end{array}$$

commutes because $f([G, G]) \subseteq [H, H]$ as $f(g^{-1}h^{-1}gh) = f(g)^{-1}f(h)^{-1}f(g)f(h)$ and so this natural transformation is well defined.

We can see that natural transformations act as morphisms between functors. As with any good morphism, we have isomorphisms which define the isomorphisms of categories.

Example 3.1.3.

1. The identity natural transformation whose morphisms are $\mathrm{id}_X : F(X) \to F(X)$ is a natural isomorphism.

2. If G is a group let G^{op} be the opposite group defined as reversing the side of group multiplication or more formally as the group which gives the category $(BG)^{op}$. This process defines a functor $(-)^{op} : \mathbf{Grp} \to \mathbf{Grp}$ which sends a homomorphism $\phi : G \to H$ to ϕ^{op} which is defined as $\phi^{op}(g) = \phi(g)$. We can see that $(-)^{op}$ is naturally isomorphic to the identity by taking the components $\eta_G : G \to G^{op}$ to send $g \in G$ to g^{-1}. The diagram commutes because $\phi^{op}(\eta_G(g)) = \phi^{op}(g^{-1}) = \phi(g^{-1}) = \phi(g)^{-1} = \eta_H(\phi(g))$.

3. Let $C, O : \mathbf{Top}^{op} \to \mathbf{Poset}$ be functors into the category of posets where C and O send a topology to its collection of closed or open sets respectively. In both cases C, O send a continuous map $f : X \to Y$ to the function $f^{-1} : C(Y) \to C(X)$ or $f^{-1} : O(Y) \to O(X)$ which sends a closed or open set $U \in Y$ to $f^{-1}(U)$ which is closed or open respectively by continuity. C and O are naturally isomorphic by the transformation η whose components η_X send an open (or closed set) to its complement which is closed (or open).

3.2 Category Equivalence

Functors act as morphisms on a category of categories. In particular this means it makes sense to speak of isomorphisms of categories. Isomorphism of categories is a very strong condition, and in practise categories can have similar structures without being isomorphic. For these we have the following notion.

Definition 3.2.1. Two categories C, D are ***equivalent*** or ***naturally equivalent***, denoted $C \simeq D$, if there exist functors $F : C \to D$ and $G : D \to C$ such that $FG \cong \mathrm{id}_D$ and $GF \cong \mathrm{id}_C$. That is to say there exists natural isomorphisms between the functor compositions and the identity functor on each category.

This definition can be compared to that of homotopy equivalence. In both cases we say that two objects are equivalent if their morphisms are equivalent to the identity. Also, similarly to how homeomorphic implies homotopy equivalence, an isomorphism of categories implies equivalence.

Lemma 3.2.2. *Let C and D be categories. Then $C \cong D$ implies $C \simeq D$*

Proof. If $C \cong D$ then there exists functors $F : C \to D$ and $F^{-1} : D \to C$ such that $FF^{-1} = \mathrm{id}_D$ and $F^{-1}F = \mathrm{id}_C$. Since the identity is naturally isomorphic to itself we see that F and F^{-1} define a natural equivalence. □

Of course we also need to show that natural equivalence is indeed an equivalence relation for this to be well defined.

Lemma 3.2.3. *Natural equivalence of categories is an equivalence relation.*

Proof. For every category C we clearly have $C \simeq C$ because $C \cong C$ by the identity transformation. If $C \simeq D$ by functors $F : C \to D$ and $G : D \to C$, then by symmetry functors $G : D \to C$ and $F : C \to D$ imply $D \simeq C$. Finally if $C \simeq D$ and $D \simeq E$ then we have the data $F_C : C \to D,\ G_C : D \to C,\ F_D : D \to E$ and $G_D : E \to D$ such that $F_C G_C \cong \mathrm{id}_D,\ G_C F_C \cong \mathrm{id}_C$ and $F_D G_D \cong \mathrm{id}_E,\ G_D F_D \cong$

id_D. Now notice that $F_D F_C : C \to E$ and $G_C G_D : E \to C$ define a natural equivalence because $F_D F_C G_C G_D \cong F_D \mathrm{id}_D G_D = F_D G_D \cong \mathrm{id}_E$ and similarly $G_C G_D F_D F_C \cong \mathrm{id}_C$. □

Checking equivalence requires the construction of two functors and two natural isomorphisms and this is not always trivial. An easier equivalent condition exists but first we need a quick definition.

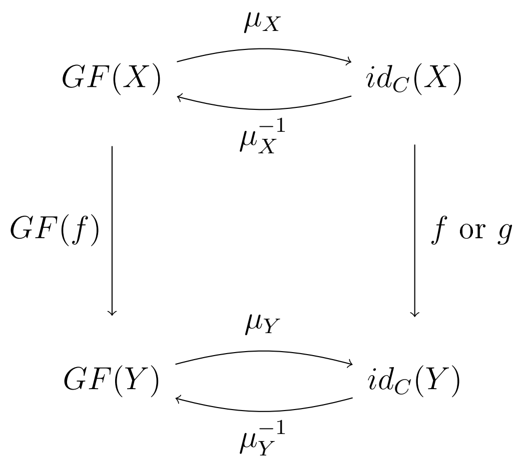
Definition 3.2.4. A functor $F : C \to D$ is ***essentially surjective*** if for every object $Y \in D$ there exists an object $F(X)$ in the image of F isomorphic to D.

This becomes part of the following criteria for equivalence.

Theorem 3.2.5. *A functor F defines a natural equivalence if and only if F is fully faithful and essentially surjective.*

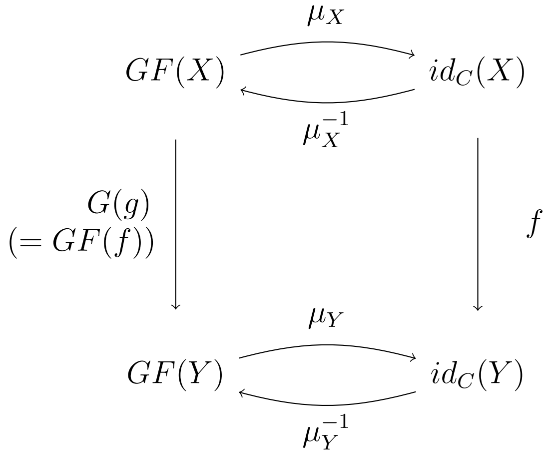
Proof. Let $F : C \to D$ and $G : D \to C$ define an equivalence of categories. Furthermore let $\eta : FG \to \mathrm{id}_D$ and $\mu : GF \to \mathrm{id}_C$ be the natural isomorphisms which gives us $FG \cong \mathrm{id}_D$ and $GF \cong \mathrm{id}_C$ respectively. If X is an object in D then η_X is an isomorphism $FG(X)$ to X. Hence, for any object X, $G(X)$ is an object in C such that $F(G(X)) \cong X$ and so F is essentially surjective.

To show F is faithful assume $f, g : X \to Y$ are morphisms in C such that $GF(f) = GF(g)$ then the following diagram commutes.



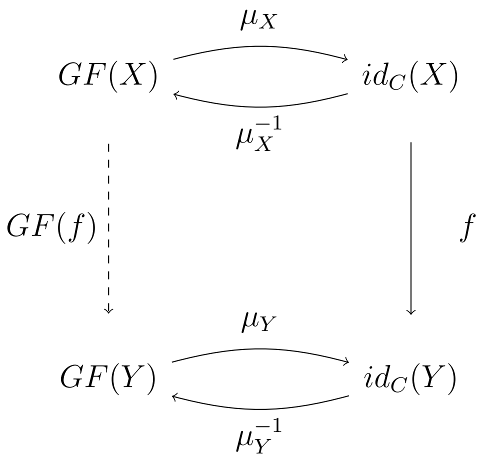
Hence, $\mu_Y^{-1} f \mu_X = \mu_Y^{-1} g \mu_X$. Since μ_X and μ_Y are bijections, they have inverses and we can cancel these terms to get $f = g$. Therefore, GF is injective and so F is injective on the morphisms between X and Y because if $F(f) = F(g)$ then $G(F(f)) = G(F(g))$. By a similar argument on FG we see that in fact both F and G are faithful.

Finally to see F is a full functor let $F(X)$ and $F(Y)$ be in the image of objects of F. Let $g : F(X) \to F(Y)$ be a morphism in D. We are required to show there exists f such that $F(f) = g$. If such an f exists then because F and G are natural equivalences, the following diagram must commute.



Hence, we know that f, if it exists, must be equal to $\mu_Y G(g)\mu_X^{-1}$. Define $f := \mu_Y G(g)\mu_X^{-1}$ then we must have $G(g) = GF(f')$ but since G is faithful then it is injective on g and $g = F(f)$. Hence we have shown that F is full.

Now for the converse. Let $F : C \to D$ be a fully faithful and essentially surjective functor. We are required to define a functor $G : D \to C$ and natural isomorphisms $\eta : FG \to \mathrm{id}_D$ and $\mu : GF \to \mathrm{id}_C$. Since isomorphism defines an equivalence relation then we can consider equivalence classes of D defined by isomorphism. Since F is essentially surjective, we can consider the collection of isomorphism classes and choose from each a representative object of C such that its image under F is an element of the isomorphism class. Let the function $R : \mathrm{ob}(D) \to \mathrm{ob}(C)$ send an object $Z \in D$ to such a choice of representative in C whose image is in its isomorphism class. By this definition objects $Z \cong W \in D$ if and only if $R(Z) = R(W)$ and $Z \cong W \cong F(R(Z))$. We define G on an object Z of D as $G(Z) = R(Z)$. If X is an object of C then because F is fully faithful and because $F(R(F(X)) = F(X)$, by Lemma 2.2.7 we have that $R(F(X)) \cong F(X)$. For each X in C choose an isomorphism μ_X from $R(F(X))$ to X and let μ be the collection of such isomorphisms. Finally for each $X \in C$ then we've seen $RF(X) = GF(X) \cong X$ so all that remains for μ to be a natural isomorphism is to define $GF(f)$ such that the following diagram commutes.



Since F is fully faithful we have that the map $F_{XY} : \hom(X, Y) \to \hom(F(X), F(Y))$

induced by F is a bijection. Hence, we define G on $F(f)$ such that $G(F(f)) = \mu_Y^{-1} f \mu_X$. In particular, for each object $Z \in D$ then $F(R(Z))$ is Z's isomorphism class representative so choose $r_Z : Z \to F(R(Z))$ be an isomorphism defining it as such. Now for every $g : Z \to W \in D$ we have $G(g) = \mu_{R(W)}^{-1} F_{R(Z)R(W)}^{-1}(r_W \circ g \circ r_Z^{-1})\mu_{R(Z)}$. By this definition our diagram commutes and μ is a well defined natural isomorphism $GF \cong \mathrm{id}_C$.

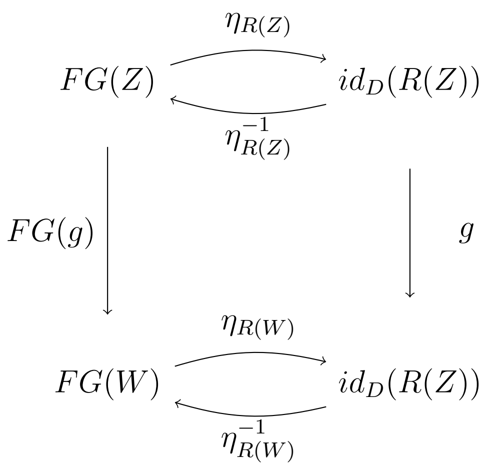
It remains to show that G is a well defined functor and also that there exists a natural isomorphism $\eta : FG \to \mathrm{id}_D$. First lets show G is well defined. If $g : Z \to W$ is a morphism in D then $G(f) = \mu_{R(W)}^{-1} F_{R(Z)R(W)}^{-1}(r_W \circ g \circ r_Z^{-1})\mu_{R(Z)}$ is a morphism $R(Z) \to R(W)$ and by definition of G we have $G(Z) = R(Z)$ and $G(W) = R(W)$ as required.

Composition is well defined because for morphisms $f : V \to W$ and $g : W \to Z$ in D we have,

$$\begin{aligned}
G(g)G(f) &= \mu_{R(Z)}^{-1} F_{R(W)R(Z)}^{-1}(r_Z \circ g \circ r_W^{-1})\mu_{R(W)}\mu_{R(W)}^{-1} F_{R(V)R(W)}^{-1}(r_W \circ f \circ r_V^{-1})\mu_{R(V)} \\
&= \mu_{R(Z)}^{-1} F_{R(W)R(Z)}^{-1}(r_Z \circ g \circ r_W^{-1}) F_{R(V)R(W)}^{-1}(r_W \circ f \circ r_V^{-1})\mu_{R(V)} \\
&= \mu_{R(Z)}^{-1} F_{R(V)R(Z)}^{-1}(r_Z \circ g \circ r_W^{-1} r_W \circ f \circ r_V^{-1})\mu_{R(V)} \\
&= \mu_{R(Z)}^{-1} F_{R(V)R(Z)}^{-1}(r_Z \circ (gf) \circ r_V)\mu_{R(V)} \\
&= G(gf).
\end{aligned}$$

To show G preserves identity morphisms take an object $Z \in D$ and we have $G(\mathrm{id}_Z) = \mu_{R(Z)}^{-1} F_{R(Z)R(Z)}^{-1}(r_Z \circ \mathrm{id}_Z \circ r_Z^{-1}))\mu_{R(Z)} = \mu_{R(Z)}^{-1} F_{R(Z)R(Z)}^{-1}(\mathrm{id}_{F(R(Z))})\mu_{R(Z)} = \mu_{R(Z)}^{-1}\mathrm{id}_{R(Z)}\mu_{R(Z)} = \mathrm{id}_{R(Z)} = \mathrm{id}_{G(Z)}$ as required.

Finally it remains to construct an natural isomorphism $\eta : FG \cong \mathrm{id}_D$. By our definitions $FG(Z) = R(Z)$ so we need to define η such that for all $g \in D$ the following diagram commutes.



Since $G(g) = \mu_{R(W)}^{-1} F_{R(Z)R(W)}^{-1}(r(W) \circ g \circ r_Z^{-1})\mu_{R(Z)}$ then $FG(g) = F(\mu_{R(W)}^{-1})(r_W \circ g \circ r_Z^{-1})F(\mu_{R(Z)})$ (notice $F(\mu_{R(Z)})$ is well defined because $\mu_{R(Z)}$ is an endomorphism) and so commutativity is equivalent to finding η such that $\eta_{R(W)}^{-1} g \eta_{R(Z)} = F(\mu_{R(W)}^{-1})(r_W \circ g \circ r_Z^{-1})F(\mu_{R(Z)})$. If we define $\eta_{R(V)}$ for an object V to be equal to $r_V^{-1}F(\mu_{R(V)})$ then, because $\eta_{R(W)}^{-1} = F(\mu_{R(W)})^{-1}(r_V^{-1})^{-1} = F(\mu_{R(W)}^{-1})r_V$ this definition allows our diagram to commute. Hence, η is a natural transformation. η^{-1} is a natural transformation similarly and hence η is a natural isomorphism and so F is a natural equivalence as required. □

It is clear that if two categories are isomorphic then they are naturally equivalent (if a functor $F : C \to D$ is an isomorphism then $FF^{-1}\mathrm{id}_D$, $F^{-1}F = \mathrm{id}_C$ so the identity natural transformation gives equivalence). It is clear that the converse is not true in general. However, natural equivalence can be thought of category isomorphism up to isomorphism of its objects. We can use the previous lemma to formalise this.

Lemma 3.2.6. *Let C, D be categories in which two objects are isomorphic if and only if they equal. Then C and D are isomorphic as categories if and only if they are naturally equivalent.*

Proof. We have already seen that category isomorphism implies equivalence in general. To see the converse let $F : C \to D$ and $G : D \to C$ define an equivalence of categories with the equivalence giving rise to the natural isomorphism $\eta : GF \to \mathrm{id}_C$. Firstly, if X is an object in C then $\eta_X(GF(X)) = X$ but since η_X is an isomorphism then $GF(X) = X$ by the definition of C. Since G gives a unique inverse of F on objects then F is a bijection on the objects.

Finally, since F is an equivalence of categories then F is fully faithful by Theorem 3.2.5 and therefore, bijective on morphisms of C. Hence, F is an isomorphism of categories. □

In fact every category is naturally equivalent to a category with no isomorphisms [6]. Identifying isomorphic objects of a category is called taking the ***skeleton*** of a category. Hence, natural equivalence can be thought of as an isomorphism of skeletons.

Now we have some understanding of category equivalence lets look at some examples.

Example 3.2.7.

1. Let $F : \mathbf{Mat}_R \to \mathbf{Vec}_R$ be a functor which sends n to R^n and a morphism $M \in \mathbf{Mat}_R$ to its corresponding linear map $T : x \mapsto Mx$. Then F defines an equivalence of categories. Certainly F is a functor of categories. To see that it defines an equivalence we check that F is both fully faithful and essentially surjective. To see that functor F is full, recall that if $T : R^n \to R^m$ is a linear transformation then there is a matrix M whose columns are given by T on each member of an ordered basis of R^n. The functor F is faithful because if two matrices M and M' act the same on a basis then they must have the same matrix entries. Finally notice that F is essentially surjective because all objects in $\mathbf{Vec}_R$ are finite and integer dimensional.

   $\mathbf{Mat}_R \simeq \mathbf{Vec}_R$ by the functor $F : \mathbf{Mat}_R \simeq \mathbf{Vec}_R$ which sends n to R^n and a morphism $M \in \mathbf{Mat}_R$ to its corresponding linear map which sends a vector x to Mx.

2. Define **1** as the category with one object and only the identity morphism. Let C be a category where $\text{ob}(C) = \{1, 2\}$ and the only morphisms are the identities and isomorphisms $1 \to 2$ and $2 \to 1$. By Theorem 3.2.5 **1** and C are equivalent categories because any functor $\mathbf{1} \to C$ is fully faithful and essentially surjective.

3. It is a famous result of Lie theory that the category of simply connected Lie groups and the category of Lie algebras are equivalent as categories by the functor taking a group to its tangent space at the identity [1].

4. Let C and D be two categories and a functor $F : C \to D$ define an equivalence. The functor $F^{op} : C^{op} \to D^{op}$ is fully faithful and essentially surjective since F has those properties and so $C^{op} \simeq D^{op}$.

3.2.1 Functor Categories

We have shown that functors have natural transformations between them. Notice that natural transformations can be composed as in Figure 3.1.

$$\begin{array}{ccccc}
F(X) & \xrightarrow{\eta_X} & G(X) & \xrightarrow{\mu_X} & H(X) \\
\Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} & & \Big\downarrow{\scriptstyle H(f)} \\
F(Y) & \xrightarrow[\eta_Y]{} & G(Y) & \xrightarrow[\mu_Y]{} & H(Y)
\end{array}$$

Figure 3.1: Composition of natural transformations.

We've also seen in Example 3.1.3 that an identity natural transformation exists. Clearly composition of natural transformations is associative so natural transformations could act as the morphisms in a category of functors. This notion gives us the following definition.

Definition 3.2.8. For categories D and C the ***functor category*** D^C has objects as all functors $F : C \to D$ and morphisms as natural transformations.

Notice that D^C is locally small if and only if C is small. Normal examples of functor categories are simply categories of some collection of functors. However, there are some other natural examples.

Example 3.2.9.

1. If C is a discrete category such that the objects are a finite set of integers 1 to n then D^C is the product category D^n. Each functor assigns a number

of C to an object in D and so each functor can be thought of as indexing D by C. The natural transformations are then the morphisms of objects $(X_1, X_2, \ldots X_n)$ to $(Y_1, Y_2, \ldots Y_n)$ taken component-wise.

2. For a group G, the category of G-sets (group actions on sets) defines the functor category $\mathbf{Set}^{BG}$.

3. Similarly the category of category of G-representations is a functor category.

4. For a category C then the presheaves $F : C^{op} \to \mathbf{Set}$ form the functor category $\mathbf{Set}^{C^{op}}$.

3.3 Representable Functors and the Yoneda Lemma

3.3.1 Representable Functors

We will start with a definition.

Definition 3.3.1. For a locally small category C with object X and functor $F : C \to \mathbf{Set}$ then a ***co-representation*** (resp. ***representation***) of F is the pair (X, η) for some natural isomorphism $\eta : \hom(X, -) \to F$ (resp. $\eta : \hom(-, X) \to F$) from the Hom functor to F. A functor F is ***co-representable*** (resp. ***representable***) if a co-representation (resp. representation) of F exists.

Some examples of representable functors are as follows.

Example 3.3.2.

1. The identity functor $I : \mathbf{Set} \to \mathbf{Set}$ is co-represented by $(\{1\}, \mathrm{im})$ where im sends an element of $f \in \hom(\{1\}, S)$ to its image $f(1)$ which clearly defines a bijection $\hom(\{1\}, S) \cong S$.

2. Most forgetful functors are co-representable. Consider the forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$ and the representation $(\mathbb{Z}, \mathrm{im}(1))$ where $\mathrm{im}(1)$ defines a bijection $\hom(\mathbb{Z}, G) \cong F(G)$ by sending an element f of the Hom set to the element $f(1)$. This is a bijection because $\mathbb{Z}$ is the free group generated by 1 so every homomorphism $\mathbb{Z} \to G$ is decided by where it sends 1. We $\mathbb{Z}$ to be a free group or the homomorphisms would impose restrictions on the morphisms between sets. Similarly the forgetful functor $\mathbf{Ring} \to \mathbf{Set}$ is co-represented by $(\mathbb{Z}[x], \mathrm{im}(x))$ and the functor $\mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$ is co-represented by $(\mathbb{R}, \mathrm{im}(1))$.

3. For a group G then the functor $F_S : BG \to \mathbf{Set}$ which sends the object $\bullet \in BG$ to the set S defines the group action of G on S. Since $\hom(\bullet, \bullet) \cong G$ as a set then F_S is co-representable if $|S| = |G|$ and F defines a bijection G to the automorphisms of S. A choice of natural transformation $\eta : \hom(\bullet, -) \to F_S$ amounts to choosing the inverse of the identity $e \in G$.

4. A presheaf is representable if it is isomorphic to $\hom(-, X)$.

5. The functor **Top** $\to$ **Set** which maps a topological space to its set of paths is co-represented by the unit interval I and the identity transformation. We can see this by noticing that $\hom(I, X)$ is exactly the set of paths.

3.3.2 Yoneda Lemma

We are now able to consider what is perhaps the most famous and important result in all of category theory [7, page 57].

Theorem 3.3.3 (Yoneda Lemma)**.** *Let C be a locally small category, $F : C \to$ **Set** be a functor, $X \in C$ be an object, and $y(X) = \hom(X, -)$ be a Hom functor. There exists a bijection γ_{FX} between the Hom set of natural transformations $\hom(y(X), F)$ and the set $F(X)$. Furthermore, if there are two functors $F, G : C \to$ **Set** with natural transformation η between them and $f : X \to Y$ is a morphism in C then both diagrams in Figure 3.2 commutes.*

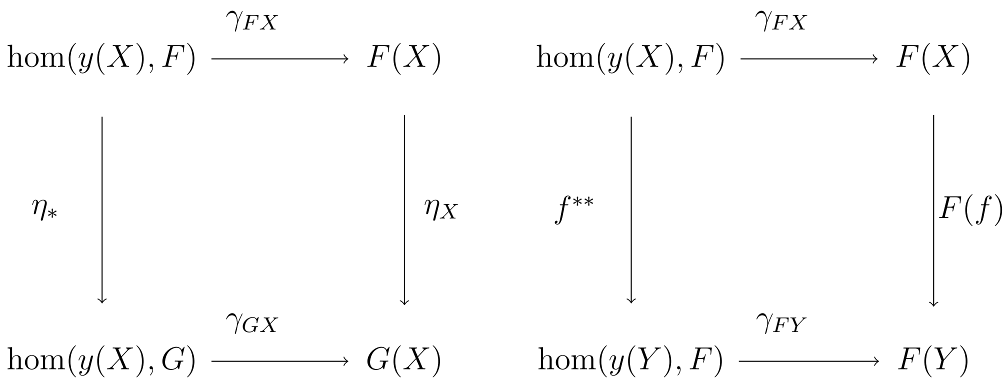


Figure 3.2: Yoneda Lemma commuting diagram.

Put differently, collection of γ_{FX}'s define a natural isomorphism γ_F between $\hom(y(X), -) : \mathbf{Set}^C \to \mathbf{Set}$ and the functor which evaluates an element of $\mathbf{Set}^C$ at X. In addition γ_{FX} define a natural isomorphism $\gamma_F : \hom(y(-), F) \to F$.

Proof. First lets show there is a bijection $\gamma : \hom(\hom(X, -), F) \to F(X)$. For a natural transformation $\mu : \hom(X, -) \to F$ let $\gamma(\mu) := \mu_X(\mathrm{id}_X)$ then we are required to show this has an inverse. I claim that the inverse of γ sends an element $q \in F(X)$ to the natural transformation $\lambda(q)$ whose component at an object Y is given by $\lambda(q)_Y : f \mapsto (F(f))(q)$. We need to show that λ does indeed define natural transformations and that λ and γ are inverse. Firstly for some $q \in F(x)$ then $\lambda(q)$ is a natural transformation if for some morphism $f : Y \to Z$ and the following diagram commutes.

$$
\begin{array}{ccc}
\hom(X,Y) & \xrightarrow{\lambda(q)_Y} & F(Y) \\
\downarrow {\scriptstyle f_*} & & \downarrow {\scriptstyle F(f)} \\
\hom(X,Z) & \xrightarrow[\lambda(q)_Z]{} & F(Z)
\end{array}
$$

To see this choose a morphism $a \in \hom(X,Y)$ and then notice that $\lambda(q)_Z(f_*)(a) = \lambda(q)_Z(f \circ a) = (F(fa))(q) = (F(f)F(a))(q) = F(f)(F(a)(q)) = F(f)\lambda(q)_Y(a)$ and so $\lambda(q)$ is a well defined natural transformation.

To see that $\lambda = \gamma^{-1}$ firstly λ is right inverse to γ because $\gamma(\lambda(q)) = \lambda(q)_X(\mathrm{id}_X) = (F(\mathrm{id}_X))(q) = \mathrm{id}_{F(X)}(q) = q$ hence $\gamma \circ \lambda = \mathrm{id}_{F(X)}$.

It remains to see $\lambda \circ \gamma = \mathrm{id}_{\hom(y(X),F)}$ so let $\mu : y(X) \to F$ be a natural transformation and we are required to check $\lambda(\gamma(\mu))$ has the same components as μ so let $f \in \hom(X,Y)$. We need to show $\lambda(\gamma(\mu))_Y(f) = \mu_Y(f)$.

$$\lambda(\gamma(\mu))_Y(f) = \lambda(\mu_X(\mathrm{id}_X))_Y(f) = (F(f))(\mu_X(\mathrm{id}_X)) = (F(f))(\mu_X))(\mathrm{id}_X)$$

Since μ is a natural transformation the following square commutes.

$$
\begin{array}{ccc}
\hom(X,X) & \xrightarrow{\mu_X} & F(X) \\
\downarrow {\scriptstyle f_*} & & \downarrow {\scriptstyle F(f)} \\
\hom(X,Y) & \xrightarrow[\mu_Y]{} & F(Y)
\end{array}
$$

Figure 3.3: Commutativity of $\lambda(q)$.

Hence, $(F(f))(\mu_X))(\mathrm{id}_X) = (\mu_Y)(f_*)(\mathrm{id}_X) = (\mu_Y)(f \circ \mathrm{id}_X) = \mu_Y(f)$ and $\lambda = \gamma^{-1}$ as required and so γ defines a bijection.

It remains to show the naturality condition of the Yoneda Lemma. For the first of the two squares we need to show $\eta_X \gamma_{FX} = \gamma_{GX}(\eta_*)$ for some natural transformation $\eta : F \to G$. Let μ be a natural transformation in $\hom(y(X), F)$ then $\eta_X \gamma_{FX}(\mu) = \eta_X(\mu_X(\mathrm{id}_X)) = (\eta_X \mu_X)(\mathrm{id}_X) = \gamma_{GX}(\eta \circ \mu) = \gamma_{GX}(\eta_*)(\mu)$ as required.

For the second of the commutating squares for morphism $f : X \to Y$ and μ, a natural transformation in $\hom(y(X), F)$ then $\gamma_{FY}(f^{**}(\mu)) = (\mu \circ f^*)_Y(\mathrm{id}_Y)$. Since μ is a natural transformation then it commutes much like $\lambda(q)$ in Figure 3.3 and we have $(\mu \circ f^*)_Y(\mathrm{id}_Y) = \mu_Y(f) = F(f)\mu_X(\mathrm{id}_X) = F(f)(\gamma_{FX}(\mu))$ so we have naturality as required. □

While the Yoneda lemma is a very abstract result, it can be applied to show powerful theorems. A notable example is the application of the Yoneda lemma to prove Cayley's theorem.

Theorem 3.3.4 (Cayley). *Every group is a subgroup of a symmetric group.*

Proof. Applying the Yoneda lemma to the functor $y(\bullet) := \hom(\bullet, -)$ we have,

$$\hom(y(\bullet), y(\bullet)) \cong \hom(\bullet, \bullet).$$

By definition $\hom(\bullet, \bullet) \cong G$ and so $G \cong \hom(y(\bullet), y(\bullet))$. Since there is only one object in BG a natural transformation $\eta : y(\bullet) \to y(\bullet)$ has only one component. Since $y(\bullet)(\bullet) = \hom(\bullet, \bullet) = G$ then a natural transformation η is just a set map $G \to G$ and hence an element of the symmetric group on elements of G. In particular the Yoneda lemma tells us that the set of natural transformations $\hom(y(\bullet), y(\bullet)) \cong G$ and so G is a subgroup of the symmetric group on the elements of G.

□

Chapter 4

Limits and Colimits

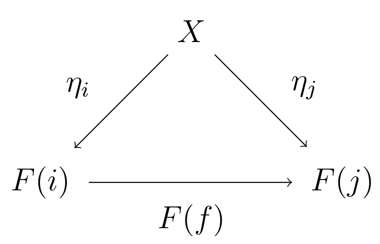
4.1 Defining Limits

When we consider the category of sets then we know that for two sets S and T then $S \times T$ is also an object in **Set**. This is also true for pairs of groups, rings and topological spaces in their respective categories. We call this a product and as with most properties that are shared between categories there is a category theoretic way to describe the product as well as other constructions. The tools we have to make this description are limits and co-limits. We will come back to the example of products later but first we need some definitions.

Definition 4.1.1. For a pair of categories C and J then a ***diagram*** is a functor $F : J \to C$ with ***index category*** J.

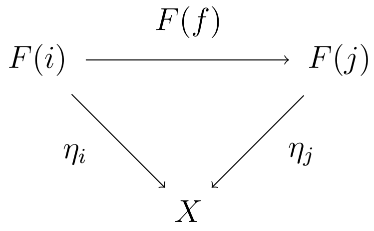
A diagram F with index category J is sometimes also called a diagram of ***shape*** J. Here we use J to describe the properties we want for our limit. For example the product of two elements is given by a discrete index category of two objects. The diagram describes which objects we choose to take the product of. Now we need to choose an object in C to be our product and a way to relate our diagram to this object. We do this through the use of ***cones***.

Definition 4.1.2. Let $F : J \to C$ be a diagram and $F_X : J \to C$ be the constant functor whose image is the object X in C. The ***cone over*** F is the natural transformation $\eta : F_X \to F$ with ***apex*** X. This means for objects $i, j \in J$ the following triangle commutes.



The cone also has a dual notion.

Definition 4.1.3. Let $F : J \to C$ be a diagram and $F_X : J \to C$ be a constant functor. The ***cocone*** is a natural transformation $\eta : F \to F_X$ with ***nadir*** X. That is, a cocone is a cone over $F^{op} : J^{op} \to C^{op}$. The naturality condition requires the following triangle commute.



Cones now allow us to properly define a limit.

Definition 4.1.4. Let $F : J \to C$ be a diagram. The ***limit*** of F is the apex X of a cone over F such that for all cones $\mu : F_Y \to F$ there exists a unique morphism $Y \to X$ where the following diagram commutes.

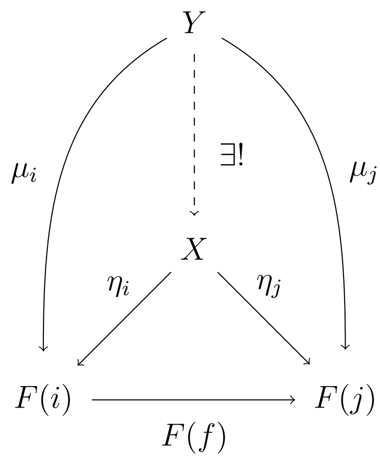
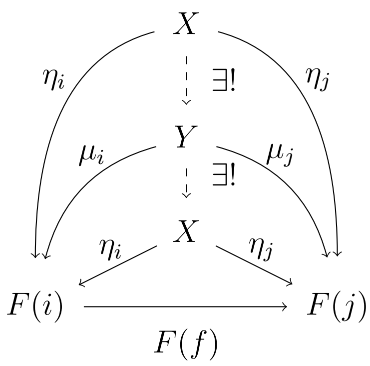


Figure 4.1: Definition of a limit.

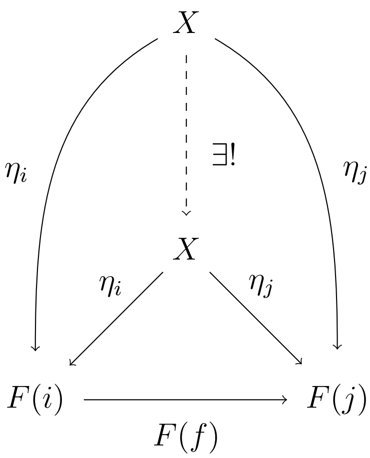
The limit of a diagram might not necessarily exist if no cones over F exist. What is more important is that our limit is defined uniquely.

Lemma 4.1.5. *If a limit exists then it is unique up to isomorphism.*

Proof. Let X and Y be limits of a diagram $F : J \to C$ with cones defined by natural transformations η and μ respectively then the following diagram commutes.



Let $g : X \to Y$ and $h : Y \to X$ be the unique maps. Because η is a cone with apex X, by definition we have that the following diagram commutes.



Certainly id_X satisfies the commutativity condition but because the map $X \to X$ has to be unique then $\mathrm{id}_X = hg$. Similarly we see that $gh = \mathrm{id}_Y$. Hence, g and h define an isomorphism $X \cong Y$. □

4.2 Examples and Properties of Limits and Colimits

Now we have the tools to define the product.

Definition 4.2.1. Let $\mathbf{2}$ be the discrete category where $\mathrm{ob}(J) = \{1, 2\}$. Define F as the diagram $F : \mathbf{2} \to C$ for some category C. Let X and Y be the objects of C such that $F(1) = X$ and $F(2) = Y$. The limit of F is the ***product*** $X \times Y$.

Because J is finite we can draw the cone over all elements in the image of J. We call this diagram the ***limit diagram***. For the product that diagram is given by Figure 4.2.

$$X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$$

Figure 4.2: Limit diagram of the product.

For every cone with apex Z, the limit requires the existence of a unique commuting map from Z into the limit diagram. The existence of this map is called the ***universal property***. We can describe this property with a diagram as in Figure 4.3.

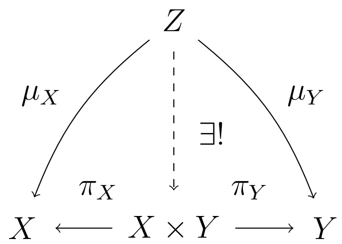


Figure 4.3: Universal property of the product.

The morphisms π_X and π_Y define the ***projection maps*** from the product to their respective objects. This notion of the product generalises to nth products simply by considering a discrete n element index category. We can also take infinite products over infinite indexing categories. The product is the usual set-wise product for all the normal categories with set structure [5]. It is worth noting however that on **Top** this limit defines the product topology and not the box topology. This is because the product topology satisfies the universal property, but is coarser than the box topology.

Another simple example of a limit is the ***terminal object*** T. This is defined as the limit of the diagram from the empty category. Explicitly, T is the object such that for every object $X \in C$ there is exactly one morphism $X \to T$. Some examples of the terminal object are as follows.

Example 4.2.2.

1. The singleton is the terminal object in **Set**. Every function into $\{*\}$ is unique and, of course, all singletons are isomorphic.

2. The trivial group is the terminal object in **Grp**.

3. The terminal object in **Ring** is the zero ring $\{0\}$.

4. In a poset category the terminal object exists if and only if there is some largest object. For example there is no terminal object in $(\mathbb{Z}, \leq)$ but -1 is the terminal object on the negative integers.

5. In **Top** and $\mathbf{Top}_*$ the one point space is the terminal object.

6. In the category of small categories the terminal object is the category **1** with one object and only the identity morphism.

7. There are no terminal object in the category of fields, because there are no homomorphisms between fields of different characteristic.

The terminal object is the unique object with exactly one morphism into it. We might ask if we can define a similar structure which gives an object which maps out to every object in the category uniquely. We call this the ***initial object*** and it is defined through the use of ***colimits***.

Definition 4.2.3. Let $F : J \to C$ be a diagram. The ***colimit*** of F is the nadir X of a cocone of F such that for all cones $\mu : F \to F_Y$ there exists a unique morphism $X \to Y$ where the following diagram commutes.

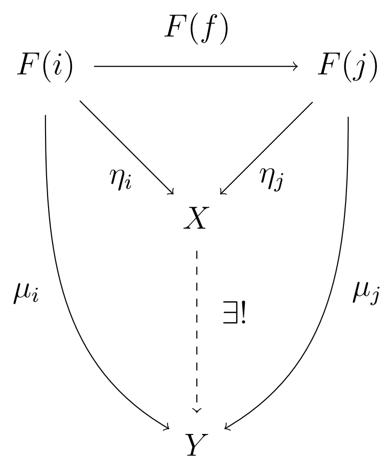


Figure 4.4: Definition of a colimit.

Much like the limit, the colimit (if it exists) is unique up to isomorphism. The initial object described above is the colimit of a diagram from the empty category. Some examples are as follows.

Example 4.2.4.

1. The empty set is the initial object in **Set** as the empty function is unique.

2. The trivial group is the initial object in **Grp**. This is the same object as the terminal object. When an object is both initial and terminal we call it the ***zero object***.

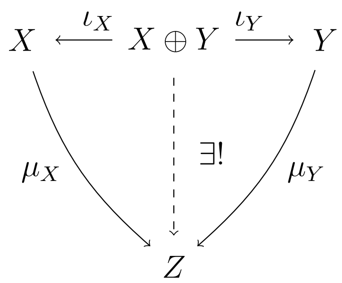
3. The initial object in **Ring** is the ring of integers $\mathbb{Z}$. The only homomorphism $f : \mathbb{Z} \to R$ sends n to $n \cdot f(1)$.

4. Similarly to the terminal object, the poset category has an initial object if and only if there is some least object.

5. In **Top** the empty set is the initial object.

6. In $\mathbf{Top}_*$ the singleton is a zero object.
7. In the category of small categories the initial object is the empty category.
8. As for terminal objects, there are no initial objects in the category of fields, because there are no homomorphisms between fields of different characteristic.

We can also consider the colimit of a diagram over the discrete category, this defines what we call the ***coproduct***. The coproduct is usually not the same as the product. The universal property of the coproduct is as follows:



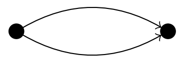
The coproducts on some different categories are as follows.

Example 4.2.5.

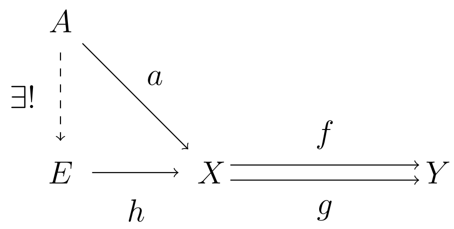
1. In the category of sets the coproduct is the disjoint union and the cone defines the inclusion maps ι.
2. The coproduct of two groups G and H defines the ***free product***. The group freely generated by elements of G and H. Clearly both G and H are subgroups of this free product.
3. The free product of two commutative groups is not necessarily commutative and hence is not the coproduct on abelian groups. The coproduct of two groups in **Ab** is the usual element-wise product. In **Ab** the finite product and finite coproduct are the same.
4. The coproduct in **Top** is the disjoint union of topologies much like in the case of the coproduct in **Set**.
5. The coproduct in $\mathbf{Top}_*$ cannot be the same as in **Top** because we would have two base points. The coproduct in based topological spaces is the disjoint union where the two based points are identified. For example the coproduct of $(S^1, 1)$ with itself is homeomorphic to a figure infinite with base point at the intersection.

Of course there are other indexing categories we can use to define limits. One of these gives rise to the object we call the kernel.

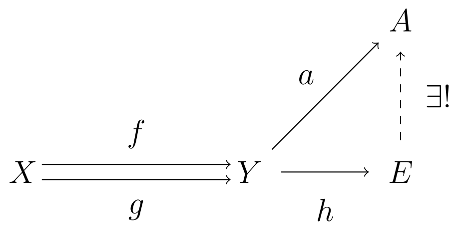
Definition 4.2.6. The ***equaliser*** of a category is the limit defined over the indexing category which has two objects and two parallel morphisms. The quiver of this indexing category is given below.



Let f and g be the image of our the morphisms of our diagram F. The universal property of the equaliser is,



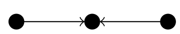
Dually we have the coequaliser which is the colimit of the same indexing category and this has universal property given by,



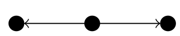
On categories with set structure the equaliser defines the object $E(f, g) := \{x \in X : f(x) = g(x)\}$. That is, the sub-object of X where f and g are equal. The map $h : E \to X$ in the above diagram is the inclusion map. On categories with a zero object like **Grp** we can define the ***kernel*** of a morphism f as the equaliser $\ker(f)$ where g is the 0 map sending X to the zero object. This defines the kernel because it defines the maximal subgroup of X where $f(\ker(f)) = g(\ker(f)) = \{e\}$. Dually when g is the 0 map then the coequaliser defines the object called the ***cokernel***. The cokernel describes the quotient group $Y/\mathrm{im}(f)$.

The final examples of limits we will discuss is that of the ***pullback*** and ***pushout***.

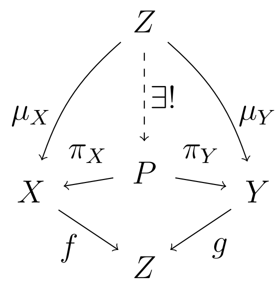
Definition 4.2.7. The ***pullback*** is the limit of a diagram with index category,



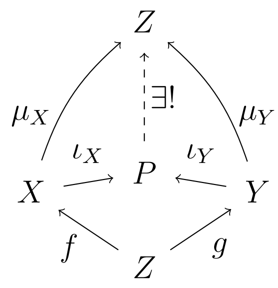
The ***pushout*** is the colimit of the diagram with the following index category.



Let the image of the morphisms in the index category be f and g then the universal property of the pullback is given by the following commuting diagram.



Similarly the universal property of the pushout is given as follows.



The pullback is also known as the fibre product. When defined on sets the pullback describes the set of elements (x, y) such that $f(x) = f(y)$. Similarly, the pullback can be defined this way on most categories with set structure like **Ring** and **Grp**. In these categories the natural transformation π is the projection. The pushout varies more over different categories but in **Set** and **Top** the pushout is the disjoint union under the equivalence relation $f(z) \sim g(z)$.

There are infinitely many more possible limits depending on our indexing category. As mentioned already these limits do not necessarily exist so it might be surprising to find that some categories, for example **Set**, actually contain all possible (small) limits. The following result allows for a condition which allows us to show this in general.

Theorem 4.2.8. *If a category has all equalisers and small products then it has all small limits.*

Proof. Let J be a small indexing category of the diagram $F : J \to C$. We need to find a cone over F which every other cone factors through uniquely. For a morphism $f : i \to j \in \hom(J)$ define the map $\mathrm{dom}(f) = i$ as the domain of f and $\mathrm{cod}(f) = j$ be the codomain. All the data of this proof is encoded in the following diagram.

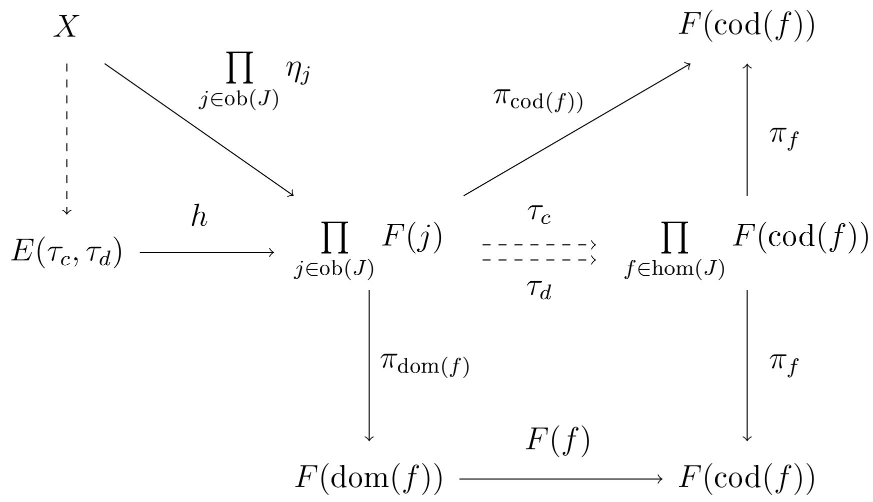


Figure 4.5: Limit as a product and equaliser.

Because all small products exist in C and J is small then the two products of the diagram are well defined. The maps π_f are the projections of the second product. We define $\pi_{\text{cod}(f)}$ (respectively $\pi_{\text{dom}(f)}$) as the projection of the object whose codomain (resp. domain) is f in the product $\prod_{X\in\text{ob}(J)} F(X)$. We define τ_c as the unique map given by the universal property of the product $\prod_{f\in\text{hom}(J)} F(\text{cod}(f))$ and the maps $\pi_{\text{cod}(f)}$. Similarly, define τ_d as the universal property of the second product and the maps $F(f) \circ \pi_{\text{dom}(f)}$. Finally, let $E(\tau_c, \tau_d)$ and h define the equaliser of these two maps. I claim $E(\tau_c, \tau_d)$ is the limit of F.

First we must show that $E(\tau_c, \tau_d)$ is the apex of a cone over F. For every pair objects i and $j \in J$ and morphism $f : i \to j$ we need to show $(\pi_j \circ h) = F(f)\circ(\pi_i\circ h)$. By definition of τ_d we know the lower square of Figure 4.5 commutes and so $F(f) \circ \pi_i = \pi_f \circ \tau_d$ and similarly for the upper triangle we have $\pi_j = \pi_f \circ \tau_c$ hence $\pi_j \circ h = (\pi_f \circ \tau_c) \circ h = \pi_f \circ \tau_d \circ h$ by the definition of the equaliser. Hence substituting for $\pi_f \circ \tau_d$, we have $(\pi_j \circ h) = F(f) \circ (\pi_i \circ h)$ as required and so we have a cone.

Finally it remains to show that for every cone over F with apex X and natural transformation η that η factors through $E(\tau_c, \tau_d)$ uniquely. The components of η form the unique map,

$$\prod_{j\in\text{ob}(J)} \eta_j : X \to \prod_{j\in\text{ob}(J)} F(j)$$

Since η is a cone over F then we know it satisfies $\tau_c \circ \prod \eta = \tau_d \circ \prod \eta$. Hence, by the definition of the equaliser we know $\prod \eta$ must factor uniquely through h and so the proof is complete.

□

Dually we can similarly show that a category has all small colimits if it has all coequalisers and small coproducts.

Chapter 5

Sheaves and the Fundamental Group

5.1 Sheaves

5.1.1 Presheaves

We already have mentioned that a presheaf is any functor $F : C^{op} \to \mathbf{Set}$. However, it is often most useful to take C as the poset category of open sets on a topological space.

Definition 5.1.1. For any topological space X, there exists a category denoted $\mathrm{Op}(X)$ whose objects are the open sets of X. For two opens $U, V \subseteq X$ then there exists a unique morphism $U \to V$ in $\mathrm{Op}(X)$ exactly when $U \subseteq V$.

We say that a functor $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ is a presheaf of the space X. For a morphism $f : U \to V$ we will denote $F(f)$ as $\rho_U^V : F(V) \to F(U)$.

Definition 5.1.2. Let $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ be a presheaf. If $U \subseteq X$ is open, the elements of the set $F(U)$ are called the ***sections*** of F over U. The sections of F over the full space X are called the ***global sections***.

Some examples of presheaves are as follows.

Example 5.1.3.

1. The ***constant presheaf*** sends each open $U \subseteq X$ to a set S. We send all morphisms to the identity, i.e. for every pair of opens $U \subseteq V$ we have $\rho_U^V = \mathrm{id}_S$.
2. Given two topological spaces X, Y then we define the presheaf $\mathcal{O}_X^Y : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ on an open $U \subseteq X$ as the set of continuous maps $U \to Y$. The morphisms ρ_U^V is the restriction from V to U.
3. We can define a presheaf much like $\mathcal{O}_X^{\mathbb{R}}$, but instead of continuous functions we send an open set U to the set of bounded functions $U \to \mathbb{R}$.

4. For a subspace $X \subseteq \mathbb{R}^n$ the functor C^r sending an open $U \subset X$ to the set of r-times differentiable functions $U \to \mathbb{R}$ (or $\mathbb{C}$) is a presheaf. The image is a subset of $\mathcal{O}_X^{\mathbb{R}}(U)$.

5. For a subspace $X \subseteq \mathbb{C}^n$ the functor C^∞ sending an open $U \subset X$ to the set of holomorphic functions $U \to \mathbb{C}$ is a presheaf. The image is a subset of $\mathcal{O}_X^{\mathbb{C}}(U)$.

6. For a set S with subset T then there is a presheaf F which sends $F(X) = S$ and $F(U) = T$ whenever $U \neq X$. For $U, V \neq X$ we have $\rho_U^V = \mathrm{id}_T$, $\rho_X^X = \mathrm{id}_S$ and ρ_U^X is the inclusion $T \to S$.

7. Similarly let $x_0 \in X$ be a point, let S be a set, and let $s \in S$ be a point. Then we define the ***indicator presheaf*** by $F(U) = S$ if $x_0 \in U$ and $F(U) = \{s\} \subseteq S$ otherwise. For opens U and V which do not contain x_0 we have $\rho_U^V = \mathrm{id}_{\{s\}}$. If U and V do contain x_0 then $\rho_U^V = \mathrm{id}_S$ if only V contains x_0 then and ρ_U^V is the inclusion $\{s\} \to S$.

8. A space X with subspace V gives rise to the ***presheaf represented by*** V H_V which sends U to the 1 point set $\{*\}$ if $U \subseteq V$ and $\emptyset$ otherwise. The maps between these sets are unique and so the image of the morphisms are decided by H_V being a functor.

9. For continuous map $p : Y \to X$ of topological spaces then there is a presheaf $\Gamma_p : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ called the ***presheaf of sections***. A section of a map $p : Y \to X$ is a (necessarily continuous) map $s : X \to Y$ such that $p \circ s = \mathrm{id}_X$. We define Γ_p as the functor which sends an open $U \subseteq X$ to the set of sections of the map which is p restricted to U.

5.1.2 Sheaves and the Sheaf Condition

In this section we will consider a specific type of presheaf called a ***sheaf***. Let $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ be a presheaf. For an open $U \subseteq X$ with open cover $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ we can define the map,

$$\prod_{\lambda \in \Lambda} \rho_{U_\lambda}^U : F(U) \to \prod_{\lambda \in \Lambda} F(U_\lambda).$$

This map gives us the two conditions on presheaves we require. The first of these is as follows

Definition 5.1.4 (Locality Condition)**.** Let $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ be a presheaf. For every open set $U \subseteq X$ and every open cover $\bigcup_{\lambda \in \Lambda} U_\lambda$, the map $\prod_{\lambda \in \Lambda} \rho_{U_\lambda}^U$ is an injection.

We call a presheaf that satisfies the locality condition a ***monopresheaf*** or in some sources a ***separated presheaf***. An important condition imposed by the locality condition is that any monopresheaf must send the empty set to a 1 point set. To see this, take the open $\emptyset$ with the empty open covering (that is $\Lambda = \emptyset$). Since $\Lambda = \emptyset$ we trivially have that $\prod_{\lambda \in \Lambda} \rho_{U_\lambda}^U$ is an injection because the product is empty but if $F(\emptyset)$ has two or more elements $s \neq s'$ then the locality condition does not hold.

The second condition we can impose on presheaves is as follows.

Definition 5.1.5 (Gluing Condition)**.** Let $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ be a presheaf. For every open cover $\bigcup_{\lambda \in \Lambda} U_\lambda$ of every open set $U \subseteq X$, the ***gluing condition*** sates the following. If there exists a collection of sections $(s_\lambda \in F(U_\lambda))_{\lambda \in \Lambda}$ which satisfy $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ for all $\alpha, \beta \in \Lambda$ then there exists a section $s \in F(U)$ such that $\prod_{\lambda \in \Lambda} \rho^U_{U_\lambda}(s) = \prod_{\lambda \in \Lambda}(s_\lambda)$.

The gluing and locality conditions together form what we call the ***sheaf condition***.

Definition 5.1.6 (Sheaf Condition)**.** A presheaf is a ***sheaf*** if it satisfies both the locality and gluing condition.

Equivalently we can re-state the sheaf condition as saying that our map $\prod_{\lambda \in \Lambda} \rho^U_{U_\lambda}$ defines the following equaliser [4, pages 65-66].

$$F(U) \xrightarrow{\prod_{\lambda \in \Lambda} \rho^U_{U_\lambda}} \prod_{\lambda \in \Lambda} F(U_\lambda) \underset{\pi_\beta}{\overset{\pi_\alpha}{\rightrightarrows}} \prod_{\alpha, \beta \in \Lambda} F(U_\alpha \cap U_\beta)$$

Where π_α and π_β are the projections whose αth and βth components are $F(U_\alpha)$ and $F(U_\beta)$ respectively followed by the restriction $F(U_\alpha \cap U_\beta)$.

For some examples of sheaves we will check whether each of the presheaves in Example 5.1.3 satisfy the sheaf condition (in the same order).

Example 5.1.7.

1. The constant presheaf F into a set S is not a sheaf because $F(\emptyset) = S$ which is not necessarily a 1 point set. If instead we define the constant presheaf such that $F(\emptyset) = \{*\}$ then F is still not a sheaf.

   In our sheaf condition we are given the map $\prod_{\lambda \in \Lambda} \rho^U_{U_\lambda}$. Since $\rho^V_{U_\lambda}$ is the identity map for all opens, then this product map is the diagonal map $S \to S^{|\Lambda|}$ by $s \mapsto (s, s, ..., s)$. This is clearly injective when $\Lambda \neq \emptyset$ and so the constant presheaf is a monopresheaf. However, this presheaf does not satisfy the gluing condition. We can take an open set U with disjoint open cover $V \sqcup W$. For every pair of section $v \in F(V)$ and $w \in F(W)$ with $v \neq w$, then $\rho^V_{V \cap W}(v) = \rho^W_{V \cap W}(w)$ because they map into a 1 point set but clearly there is no s such that $(s, s) = (v, w)$. If and only if S is a 1 point set is the constant presheaf a proper sheaf.

2. The presheaf $\mathcal{O}_X^Y$ is a sheaf. The sheaf condition says, that given some open cover, if continuous functions agree on the overlap of these opens (the $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ condition) then there is a unique continuous function which glues the continuous functions on the open sets together. This is why we call Definition 5.1.5 the gluing condition.

3. Bounded functions do not give rise to sheaves in general. This is because we cannot glue together bounded functions and guarantee a bounded function. To make this precise, consider the the bounded functions $\mathbb{R} \to \mathbb{R}$ given by $F(\mathbb{R})$ then the intervals of the form $(n, n+2)$ cover $\mathbb{R}$ and the function $f(x) = x$ is bounded on all of these intervals. Furthermore, these functions are equal on their intersections but f is not bounded on all of $\mathbb{R}$ so this fails the gluing condition.

4. Much like with $\mathcal{O}_X^Y$, the presheaves of r-times differentiable are sheaves.

5. The presheaves of holomorphic functions are sheaves.

6. The indicator presheaf is a sheaf. If U is an open with covering $\{U_\lambda\}_{\lambda \in \Lambda}$ then if $x_0 \notin U$ we have $\rho^U_{U_\lambda}(s) = \rho^U_{U_\lambda}(s')$ because these maps are both identity. If instead $x_0 \in U$ then at least one U_λ in the open covering contains x_0. Hence, our product map sends every section $s \in F(U)$ to a tuple with s is in the λth position and so this map must be injective. To see that this satisfies the gluing condition, notice that if a collection of sections satisfy $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ for all $\alpha, \beta \in \Lambda$, then for every pair U_α and U_β which both contain x_0 we have $s_\alpha = s_\beta$. If neither open contain x_0 then $\rho^U_{U_\alpha} = \rho^U_{U_\beta} = \{*\}$ on all sections and so $s = s_\alpha = s_\beta$ satisfies the gluing condition. Notice if there is a pair U_α and U_β such that U_α contains x_0 but U_β does not then $s_\alpha = *$ and so we can take $s = * \in S$ as the section which satisfies the gluing condition.

7. The presheaf H_V is a sheaf. Knowing this we will call it the ***sheaf represented by*** V.

8. The presheaf of sections is indeed a sheaf because of how we can glue continuous functions much like on the sheaf $\mathcal{O}_X^Y$. We therefore refer to this presheaf instead as the ***sheaf of sections***.

5.2 Stalks

Let us consider the behaviour of a sheaf at a particular point in our topology. Because points are often not open sets the best we can do is to consider the collection of opens around a point.

Definition 5.2.1. Let F be a presheaf of a topological space X. Let x be a point of X and $\{U_\lambda\}_{\lambda \in \Lambda}$ be the collection of open sets of X containing the point x. Let $\amalg_{\lambda \in \Lambda} F(U_\lambda)$ denote the set of pairs (U_λ, s) for $s \in F(U_\lambda)$. Define an equivalence relation on this set such that $(U, s) \sim (V, s')$ if there exists an open neighbourhood

W of x contained in $U \cap V$ such that $\rho^U_W(s) = \rho^V_W(t)$. A ***stalk*** F_x of a presheaf F at x is the set $\amalg_{\lambda\in\Lambda} F(U_\lambda)/\sim$ and a ***germ*** is the equivalence class of a section s_x under this relation.

If we treat $\{U_\lambda\}_{\lambda\in\Lambda}$ as a poset category then the stalk can be defined as the colimit of the functor $\{U_\lambda\}^{op}_{\lambda\in\Lambda} \to \mathbf{Set}$ [10]. The stalk is effectively taking sections to be equivalent if they are equal on some small open neighbourhood.

Example 5.2.2. Take the presheaf F of analytic functions on $\mathbb{R}$ and consider the stalk at 0. Two sections (analytic functions) will be equivalent in a stalk only if they're the same function on some small open neighbourhood of 0. This is only satisfied if all derivatives of our sections are the same at 0 and hence, only if they have the same Taylor expansion at 0.

Example 5.2.3. Consider the sheaf of sections F of the map $e^{ix} : \mathbb{R} \to S^1$. The sections s of an open $U \subseteq S^1$ by definition must satisfy $e^{ix} = e^{s(e^{ix})}$ on U. This is only satisfied on an open interval by $s : e^{ix} \mapsto x + 2\pi n$ for $n \in \mathbb{Z}$ and so these are the sections of U for each non-empty U. The equivalence relation defined by the stalk says that two sections $s(e^{ix}) = x + 2\pi n$ and $t(e^{ix}) = x + 2\pi m$ on opens V and U are equivalent if and only if they are the same map when restricted to the intersection $V \cap U$, if this intersection is non-empty this is precisely when $n = m$ and so the stalk F_z at some point z is simply all maps which send $e^{ix} \in S^1$ to $x + 2\pi n$.

5.3 The Étalé Space, Sheafification, and Connectedness

5.3.1 Étalé Spaces

We have found plenty of examples of presheaves which are not sheaves. There exists a process called ***sheafification*** which allows us to generate a sheaf from a presheaf. The étalé space is the tool we use to make this construction.

Definition 5.3.1. For a presheaf $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$, the ***étalé space*** is the disjoint union of all stalks,

$$\text{Ét}(F) := \bigsqcup_{x\in X} F_x.$$

For every section s of all opens $U \subseteq X$ define the collection of maps $\sigma_s : U \to \text{Ét}(F)$ which each act on U by $\sigma_s : x \mapsto s_x \in \text{Ét}(F)$. The topology given to Ét(F) is the final topology on the collection of maps σ_s. That is, Ét(F) has the finest topology such that these maps are all continuous.

We can describe the category of presheaves as objects with natural transformations as morphisms. We can just as easily describe the category of étalé spaces as a subcategory of **Top**. In this context we see that Ét is a functor between these two categories.

There is an obvious map from an étalé space to back to the topological space on which the presheaf is defined. We denote this map $\pi_F : \text{Ét}(F) \to X$ and it sends every element of a stalk $F_x \mapsto x \in X$. Notice that the maps σ_x are all sections of π_F because $\pi_F \circ \sigma_s(x) = \pi_F(s_x) = x$. Another useful property of π_F is the following.

Definition 5.3.2. A ***local homeomorphism*** is a continuous map $f : X \to Y$ such that for every point $x \in X$, there exists a local neighbourhood $x \in U$ such that f restricted to U defines a homeomorphism $U \to f(U)$. Two spaces are ***locally homeomorphic*** if there exists a local homeomorphism between them.

Notice that, as one might expect, every homeomorphism $f : X \to Y$ is a local homeomorphism because the domain X defines a neighbourhood of every point on which f is a homeomorphism. Notably, a local homeomorphism is the definition required to describe a manifold.

Definition 5.3.3. A ***manifold*** is a topological space which is locally homeomorphic to Euclidean space.

Local homeomorphisms are important to us because the map $\pi_F : \text{Ét}(F) \to X$ is a local homeomorphism. In fact, the definition of the étalé space is often taken as the pair of a topological space and a local homeomorphism [9, page 18].

By definition we know that all sheaves are presheaves. We can also create a sheaf 'generated' from a given presheaf using the tools we've discussed.

Definition 5.3.4. For a presheaf $F : \text{Op}(X)^{op} \to \textbf{Set}$ the ***sheafification*** of F is the sheaf of sections of the map $\pi_F : \text{Ét}(F) \to X$. We denote the sheafification of F as $aF := \Gamma_{\pi_F} : \text{Op}(X)^{op} \to \textbf{Set}$. We call the sheaf aF the ***associated sheaf***.

The sheafification satisfies the following universal property.

Lemma 5.3.5 (Universal property of sheafification)**.** *Let F be a presheaf, G be a sheaf, and a natural transformation $\eta : F \to G$. There exists a unique natural transformation $aF \to G$ such that the following diagram commutes.*

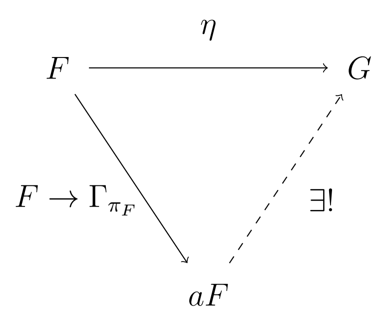


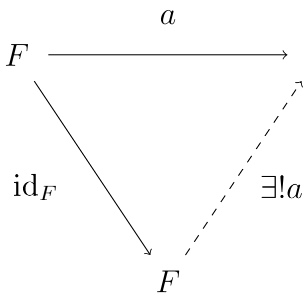
Figure 5.1: The universal property of sheafification.

Proof. This is shown Theorem 4.2 of [9]. □

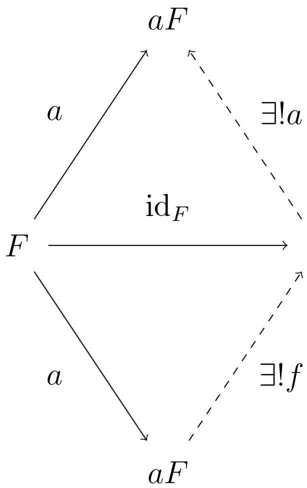
The sheafification of a presheaf is certainly a sheaf because every presheaf of sections is a sheaf. The universal property shows us that $aF \cong F$ if F is a sheaf.

Lemma 5.3.6. *Let $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ be a sheaf. The sheafification aF is naturally isomorphic to F.*

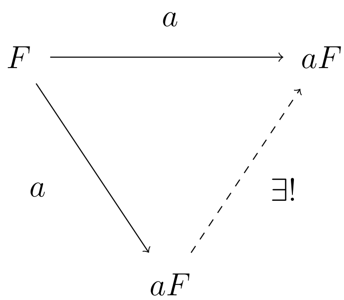
Proof. Notice that F with the identity satisfies the universal property of Figure 5.1. That is, let G be a sheaf with natural transformation $a : F \to G$. Since $f = a$ is the only map to satisfy $f \circ \mathrm{id} = a$, then a is the unique map which allows the following diagram to commute.



In particular take $G = aF$ and a as the map $F \to \Gamma_{\pi_F}$. Since aF satisfies the universal property by Lemma 5.3.5 we know the following diagram commutes.

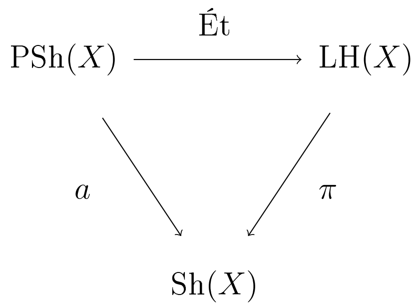


Furthermore, since aF satisfies the universal property we know there is a unique map which commutes in the diagram,



Since id_{aF} commutes in this diagram it must be the unique map. In particular, we know $af = \mathrm{id}_{aF}$ and by a similar argument we see $fa = \mathrm{id}_F$. Hence, a is a natural isomorphism and so $F \cong aF$. □

Since $F \cong aF$ when F is a sheaf, we now can see why we use the term 'section' to describe both the sections of a map, and elements of $F(U)$. In a sheaf the sections of π_F over U are exactly the sections in $F(U)$. In fact the construction of π_F defines an equivalence of the category of local homeomorphisms of a space X and the category of sheaves over X [4, Theorem 6.2]. Furthermore our map $a : F \to aF$ defines a functor from the category of presheaves over X to the category of sheaves on X. Finally, the construction of the étalé space provides a functor from presheaves on X to the local homeomorphisms π on X. This relationship is described by the following diagram.



An illustrative example of sheafification of a presheaf is the following.

Example 5.3.7. The constant presheaf $F : \mathrm{Op}(X)^{op} \to \mathbf{Set}$ where $F(U) = S$ for all opens U is not a sheaf. The associated sheaf of F is $aF = \Gamma_{\pi_X}$ where π_X is the projection $\text{Ét}(X) \to X$. We need to find how aF acts on the opens of X. Consider the set $aF(\emptyset)$. We can observe that every section $\sigma : \emptyset \to \text{Ét}(X)$ is the same function because there are no elements in $\emptyset$ and so we can see that $aF(\emptyset)$ is a 1 point set containing only the empty function.

Now we consider non-empty U. First let us consider the étalé space of F. Notice that for any point x, the stalk is $F_x = S$. Hence, the étalé space is the disjoint union of S for each point in X and so is isomorphic as a set to $S \times X$. We require the $S \times X$ to have the final topology on the maps $\sigma_s : x \mapsto s_x$. This is achieved by giving the X component of the product the topology of X and S the discrete topology. We shall denote this topological space as $S^\delta \times X$. Now to calculate the sections of π_F, notice that $\pi_F \circ \sigma(x) = x$ precisely when $\sigma(x) = (s, x) \in S^\delta \times X$. If $U \subseteq X$ is a connected open set, then by the requirement of continuity, the sections of π_F on U are simply the maps σ_s which send x to (s, x) for some constant $s \in S$. However, if U is a union of n connected components then a section σ can map each connected component to a different $s \in S$. So each section of π_F over U is equivalent to a choice of $s \in S$ for each connected component in U and hence $aF(U) \cong S^n$.

Since this sheaf is generated by the constant presheaf we call it the ***constant sheaf***. The constant sheaf on a set S is denoted F_S.

We have a clear connection between sheaves and the collection of connected components of a topological space. For simplicity in exploring this we will consider spaces where path connected and connected are equivalent properties. Such a type of space is the following.

Definition 5.3.8. A topological space X is ***locally contractible*** if for every point $x \in X$ and every open neighbourhood U of x there exists a contractible open subset $V \subseteq U$.

Connected and path connected are equivalent properties in a locally contractible space. This is because if an open U is connected, it cannot be the union of two or more disjoint opens. Hence, any two points in U, can be connected by a series of contractible open sets. These contractible opens define a path between our two points by describing paths from the contraction point of each open to a point in their intersections.

Examples of locally contractible spaces include Euclidean space, n-spheres and projective space and all manifolds. To describe connectedness in a topological space we will use the following map.

Definition 5.3.9. Let X be a topological space and $\sim$ be an equivalence relation where $x \sim y$ when there is a path connecting x and y. The ***0th homotopy set*** $\pi_0 : X := X/\sim$ sends each point $x \in X$ to its equivalence class under $\sim$.

For example, if X is path connected then $\pi_0(X) = \{0\}$. If X has the discrete topology then $\pi_0 X = X$. Of course on a locally contractible space we could equivalently say that $\pi_0 X$ is the set of the connected components of X.

Theorem 5.3.10. *Let X be a locally contractible space X. Let the functor F : **Set** $\to$ **Set** send a set S to the to the set of global sections of the constant sheaf $F_S : \mathrm{Op}(X)^{op} \to$ **Set**. The functor F is represented by $\pi_0(X)$.*

Proof. Recall that in Example 5.3.7 we saw that the constant sheaf F_S is a functor which sends an open set U to a copy of S for each connected components of U. In a locally contractible space we know that the set of connected components is given by $\pi_0(U)$. Hence, we when we consider the global sections of F_S, we have the following,

$$F_S(X) \cong S^{|\pi_0(X)|} \cong \hom(\pi_0(X), S)$$

In particular this means that $F : S \mapsto F_S(X)$ is equivalent to the functor $\hom(\pi_0(X), -)$. □

In particular if we take $\mathrm{id}_{\mathbf{Set}}$ to be the identity functor of **Set** and then by the Yoneda lemma we have $\hom(F, \mathrm{id}_{\mathbf{Set}}) \cong \hom(\hom(\pi_0(X), -), \mathrm{id}_{\mathbf{Set}}) \cong \mathrm{id}_{\mathbf{Set}}(\pi_0(X)) = \pi_0(X)$. This means that complete knowledge of the the constant sheaves on a locally contractible topological space X tells us exactly how π_0 acts on X. As such, we can define π_0 through purely categorical means.

Conclusion

Category theory takes familiar structures and extracts their most basic features. By studying categories and their morphisms we can use general properties to show specific and often surprising results. We have looked at a variety of categories, functors, natural transformations, and limits. All of these allow us to describe shared properties of different categories.

We have proved the Yoneda lemma, a surprisingly powerful result. In this thesis we showed the Yoneda lemma is applicable to both Cayley's theorem and the connectedness of a topological space.

For a reader further interested in studying category theory I would highly recommend Emily Riehl's "Category Theory in Context" [7]. In particular, the study of adjoint functors is an important topic that we did not have time to cover here.

We have taken functors and used them to define sheaves. These tools have then been applied to construction of the 0th homotopy set π_0. A similar process allows the construction of the fundamental group π_1 through the use of what are called "locally constant sheaves". In fact, sheaves can be used to define all of the homotopy groups as proven in [11].

Bibliography

[1] Roger W. Carter et al. "Lie theory". In: *Lectures on Lie Groups and Lie Algebras*. London Mathematical Society Student Texts. Cambridge University Press, 1995, pp. 76–81. DOI: 10.1017/CBO9781139172882.012.

[2] Samuel Eilenberg and Saunders Mac Lane. "General Theory of Natural Equivalences". In: *Transactions of the American Mathematical Society* 58.2 (1945), pp. 246–248. ISSN: 00029947. URL: http://www.jstor.org/stable/1990284.

[3] Saunders Mac Lane. "Categories, Functors, and Natural Transformations". In: *Categories for the Working Mathematician*. New York, NY: Springer New York, 1971, p. 18. ISBN: 978-1-4612-9839-7. DOI: 10.1007/978-1-4612-9839-7_2. URL: https://doi.org/10.1007/978-1-4612-9839-7_2.

[4] Saunders Mac Lane and Ieke Moerdijk. "Sheaves of Sets". In: *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. New York, NY: Springer New York, 1994, pp. 65–105. ISBN: 978-1-4612-0927-0. DOI: 10.1007/978-1-4612-0927-0_4. URL: https://doi.org/10.1007/978-1-4612-0927-0_4.

[5] Tom Leinster. "Limits". In: *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2014, pp. 107–112. DOI: 10.1017/CBO9781107360068.007.

[6] Monique Pavel. "Skeletal categories". In: *Pattern Recognition* 11.5 (1979), pp. 325–327. ISSN: 0031-3203. DOI: https://doi.org/10.1016/0031-3203(79)90042-6. URL: https://www.sciencedirect.com/science/article/pii/0031320379900426.

[7] Emily Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017. ISBN: 9780486820804. URL: https://books.google.co.uk/books?id=6B9MDgAAQBAJ.

[8] Michael A. Shulman. *Set theory for category theory*. 2008. eprint: arXiv:0810.1279.

[9] B. R. Tennison. "Sheaves and sheaf spaces". In: *Sheaf Theory*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1975, pp. 14–30. DOI: 10.1017/CBO9780511661761.004.

[10] *The Stacks project*. 2022. URL: https://stacks.math.columbia.edu/tag/0078 (visited on 03/17/2022).

[11] Bertrand Toën. "Le problème de la schématisation de Grothendieck revisité". In: *Épijournal de Géométrie Algébrique* Volume 4 (Oct. 2020). DOI: 10.46298/epiga.2020.volume4.6060. URL: https://epiga.episciences.org/6822.